[FORMALITY, REDUCTION] = 0 ? FIRST STEP

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→ Reduction (M,π)

$\Phi : G\times M \to M$ Poisson action
allowing $J : M \to \mathfrak{g}^*$
} Hamiltonian

$0 \in \mathfrak{g}^* \Rightarrow C := J^{-1}(\{0\})$ closed embedded submfld

$M_{red} := C/G$ Φ proper & free $\Rightarrow$ M_{red} Poisson mfld

→ Formality

$T_{poly}(M) \xrightarrow{F} D_{poly}(M)$ $\Rightarrow$ every Poisson mfld admits canonical quantization

π Poisson iff $[\![\pi,\pi]\!] = 0$ $\overset{F}{\longleftrightarrow}$ $\mu = m + M$ associative iff M is MC element

Aim :

$$\begin{tikzcd}
T_{poly}(M) \arrow[r, "F"] \arrow[d, "?"'] \arrow[dr, phantom, "\circlearrowright"] & D_{poly}(M) \arrow[d] \\
T_{poly}(M_{red}) \arrow[r, "F"] & D_{poly}(M_{red})
\end{tikzcd}$$

$$M = C \times \mathfrak{g}^*$$

Today : classical side

$$\begin{tikzcd}
\left(T_{\mathfrak{g}}(M), \lambda, [-J,\cdot], [\cdot,\cdot]\right) \arrow[rr, "R"] \arrow[d] & & \left(T_{poly}(M_{red}), \overset{\text{no curvature}}{0}, 0, [\cdot,\cdot]\right) \\
T_{Tay}(C \times \mathfrak{g}^*) \arrow[urr, "\text{quis}"] & & T_{Cart}(C) \arrow[ll, "\iota"] \arrow[u, "p"']
\end{tikzcd}$$

Remember : (M, ω)

$$\begin{tikzcd}
 & C \arrow[dl, "\iota"'] \arrow[dr, "p"] & \\
M & & M_{red}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (S\mathfrak{g}^* \otimes \Omega(C))^G & \\
(S\mathfrak{g}^* \otimes \Omega(M))^G \arrow[ur, "\iota^*"] & & \Omega(M_{red}) \arrow[ul, "p^*"']
\end{tikzcd}$$

Cartan model

1. Equivariant Multivector fields

$$G \times M \xrightarrow{\Phi} M \quad \text{Lie group action}$$

<u>Def. 1.1</u>

$$T^{\cdot}_{\mathfrak{g}}(M) := \bigoplus_{2i+j=k} \left(S^{i}\mathfrak{g}^* \otimes \Gamma^\infty(\Lambda^{k+1} TM)\right)^G$$

$$d = 0$$

$$[\alpha \otimes X, \beta \otimes Y]_{\mathfrak{g}} = \alpha \vee \beta \otimes [X, Y]$$

in terms of polynomial maps : $\mathfrak{g} \longrightarrow T^k_{poly}(M)$

$$[X,Y]_{\mathfrak{g}}(\xi) = [\![X(\xi), Y(\xi)]\!]$$

We have a canonical map $\lambda : \mathfrak{g} \to T^0_{poly}(M) \;:\; \xi \mapsto \xi_M$ fund v.f. of Φ

central $\Rightarrow$ $T^\cdot_{\mathfrak{g}}(M)$ curved Lie alg curvature $-\lambda$

<u>Lemma 1.2</u>

Curved MC elements of $T^\cdot_{\mathfrak{g}}(M)$ equivalent to Poisson str + J

"Proof"

$P \in T^1_{\mathfrak{g}}(M)$ $\qquad -\lambda + \frac{1}{2}[P,P]_{\mathfrak{g}} = 0$

$P = \pi + J \quad \in (T^1_{poly}(M))^G \oplus (\mathfrak{g}^* \otimes T^{-1}_{poly}(M))^G$

$$\frac{1}{2}[P,P]_{\mathfrak{g}}(\xi) = \underbrace{\frac{1}{2}[\pi,\pi]}_{= 0} + \underbrace{[\pi, J_\xi]}_{= \{\cdot, J_\xi\}} + \underbrace{[J_\xi, J_\xi]}_{= 0} = \lambda(\xi)$$

$\square$

2. Taylor expansion

$$M = G \times \mathfrak{g}^*$$

$$T^k_{poly}(G \times \mathfrak{g}^*) \cong \bigoplus_{i+j=k} C^\infty(G \times \mathfrak{g}^*) \otimes \left(\Lambda^i \mathfrak{g}^* \otimes T^j_{poly}(G)\right)$$

First we define

$$T_{\mathfrak{g}^*} : C^\infty(G \times \mathfrak{g}^*)^G \longrightarrow \prod_i \left(S^i \mathfrak{g} \otimes C^\infty(G)\right)^G \qquad \alpha_i \; e^i \text{ coord on } \mathfrak{g}^*$$

$$f \longmapsto \sum_I \frac{1}{I!} e_I \otimes \iota^* \frac{\partial f}{\partial \alpha_I}$$

- $T_{\mathfrak{g}^*}$ equivariant
- restrict to G-inv
- we can extend

$$T^k_{poly}(G \times \mathfrak{g}^*) \longrightarrow \prod_i \left(S^i \mathfrak{g} \otimes \Lambda \mathfrak{g}^* \otimes T_{poly}(G)\right)$$

Def 2.1

$$\left(S\mathfrak{g}^* \otimes T_{poly}(\mathbb{C}\times\mathfrak{g}^*)\right)^G \xrightarrow{T_{\mathfrak{g}^*}} \underbrace{\left(S\mathfrak{g}^* \otimes \prod_i \left(S^i\mathfrak{g} \otimes \Lambda\mathfrak{g}^* \otimes T_{poly}(\mathbb{C})\right)\right)^G}_{T_{Tay}(\mathbb{C}\times\mathfrak{g}^*)}$$

$\rightsquigarrow$ DGLA

Lemma 2.2 $\quad T_{\mathfrak{g}^*}$ is DGLA morph.

$$\lambda = e^i \otimes (e_i)_M \in T^2_{\mathfrak{g}}(M)$$

Corollary 2.3

$$\left(T^\bullet_{\mathfrak{g}}(M), \lambda, -[J,\cdot], [\cdot,\cdot]\right) \longrightarrow \left(T_{Tay}(\mathbb{C}\times\mathfrak{g}^*), \lambda, -[J,\cdot], [\cdot,\cdot]\right)$$

is a morphism of curved lie algs

π_{KKS} canonical element in T_{Tay}

3. Cartan model

Compute cohomology of T_{Tay}

Prop. 3.1

$$\left(\prod_{i=0}^{\infty} \left(S^i \mathfrak{g} \otimes T_{poly}(\mathbb{C}) \right)^G, 0, [\cdot,\cdot] \right) \xrightarrow{\iota} \left(T_{Tay}(\mathbb{C} \times \mathfrak{g}^*), [\partial,\cdot], [\cdot,\cdot] \right)$$

!

$$\partial := id \otimes i_s(e^i) \otimes id \otimes (e_i)_{\mathbb{C}} \wedge .$$

DGLA

$$T_{cart}(\mathbb{C}) = \left(\prod_i \left(S^i \mathfrak{g} \otimes T_{poly}(\mathbb{C}) \right)^G, 0, [\cdot,\cdot] \right)$$

- Note that we have a canonical DGLA map

$$p : (T_{cart}(C), \partial, [\cdot,\cdot]) \longrightarrow (T_{poly}(M_{red}), 0, [\cdot,\cdot])$$

Prop. 3.1 p is quis.

Find extension of $[-J, \cdot]$ to make ι a quis wrt ∂

Prop. 3.2

The map $[\pi_{KKS}, \cdot]$ is a well-def differential and the inclusion

$$\iota : (T_{cart}(C), \partial, [\cdot,\cdot]) \longrightarrow (T_{Tay}(C \times g^*), [\pi_{KKS} - J, \cdot], [\cdot,\cdot])$$

Thm. 3.3 $\exists$ L_∞-quis

$$\begin{tikzcd}
(T_{Tay}(C \times g^*), [\pi_{KKS} - J, \cdot], [\cdot,\cdot]) \arrow[rr] & & (T_{poly}(M_{red}), 0, [\cdot,\cdot]) \\
& (T_{car}(C), \partial, [\cdot,\cdot]) \arrow[ul, "\iota"] \arrow[ur, "p"'] &
\end{tikzcd}$$

Formal setting :

$$
\begin{tikzcd}
\left(T_{Tay}(G\times\mathfrak{g}^*)[[\hbar]], [\hbar\pi_{KKS}-J, \cdot]\right) \arrow[rr] & & \left(T_{poly}(M_{red})[[\hbar]], 0, [\cdot,\cdot]\right) \\
& \left(T_{cont}(G)[[\hbar]], \hbar\partial\right) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Conclusion :

$$T_{\mathfrak{g}}(M)[[\hbar]] \xrightarrow{R} T_{poly}(M_{red})[[\hbar]]$$

curved L_∞ morphism

Remark : reduction procedure of M-W coincides with the one via R [for MC elements of the form $\hbar\pi \in \hbar T_{\mathfrak{g}}(M)[[\hbar]]$]

$\pi \in T^1_{poly}(M)$