

[FORMALITY, REDUCTION] = 0 ? FIRST STEP

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→ Reduction (M, π)

$\Phi: G \times M \rightarrow M$ Poisson action } Hamiltonian
allowing $J: M \rightarrow \mathfrak{g}^*$

$0 \in \mathfrak{g}^* \Rightarrow C := J^{-1}(\{0\})$ closed embedded submfld

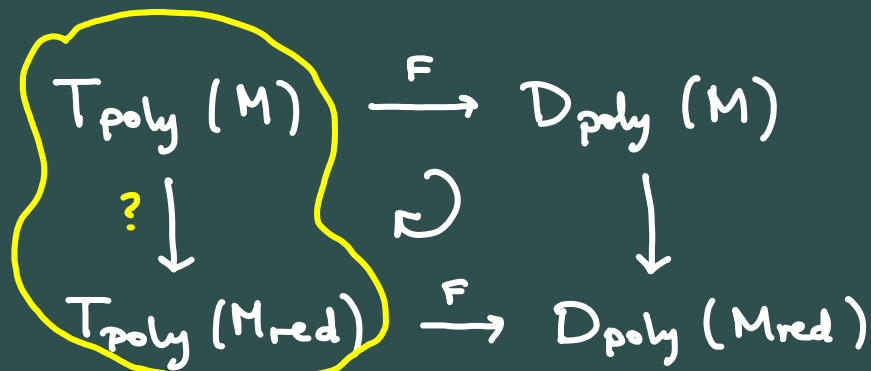
$M_{\text{red}} := C/G$ Φ proper & free $\Rightarrow M_{\text{red}}$ Poisson mfld

→ Formality

$T_{\text{poly}}(M) \xrightarrow{F} D_{\text{poly}}(M) \Rightarrow$ every Poisson mfld admits
canonical quantization

π Poisson iff $[[\pi, \pi]] = 0$ \xleftrightarrow{F} $\mu = \mu + M$
associative iff M is MC element

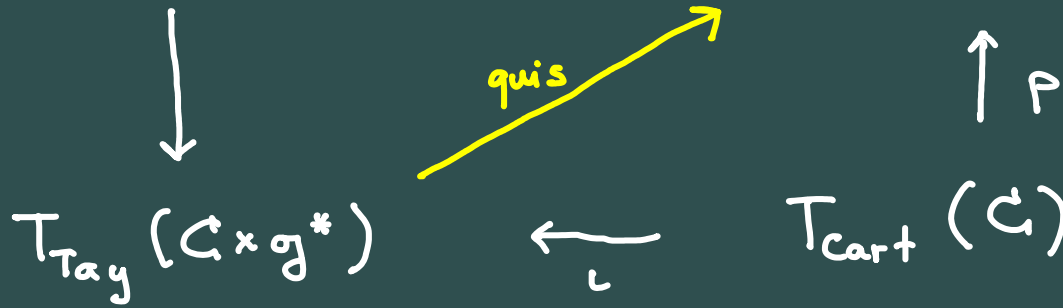
Aim:



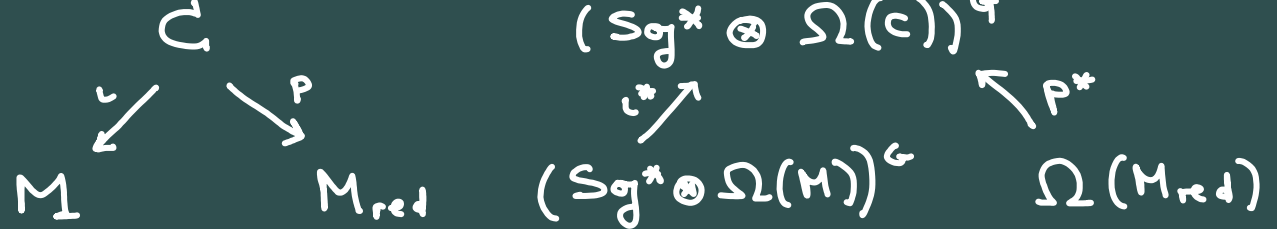
Today : classical side

$$M = \mathbb{C} \times \mathfrak{g}^*$$

$$(T_{\mathfrak{g}}(M), \lambda, [-\cdot, \cdot], [\cdot, \cdot]) \xrightarrow{R} (T_{\text{poly}}(M_{\text{red}}), \overset{\text{no curvature}}{\downarrow} 0, 0, [\cdot, \cdot])$$



Remember : (M, ω)



Cartan model

1. Equivariant Multivector fields

$$G \times M \xrightarrow{\Phi} M \quad \text{Lie group action}$$

Def. 1.1

$$T_{\mathfrak{g}}^i(M) := \bigoplus_{2i+j=k} (S^i \mathfrak{g}^* \otimes \Gamma^{\infty}(\wedge^{k+1} TM))^G$$

$$d = 0$$

$$[\alpha \otimes X, \beta \otimes Y]_{\mathfrak{g}} = \alpha \vee \beta \otimes [X, Y]$$

in terms of polynomial maps : $\mathfrak{g} \longrightarrow T_{\text{poly}}^k(M)$

$$[X, Y]_{\mathfrak{g}}(\xi) = [[X(\xi), Y(\xi)]]$$

We have a canonical map $\lambda : \mathfrak{g} \longrightarrow T_{\text{poly}}^0(M) : \xi \mapsto \xi_M$ fund v.f. of Φ

central $\Rightarrow T_{\mathfrak{g}}^1(M)$ curved Lie alg curvature $-\lambda$

Lemma 1.2

Curved MC elements of $T_{\mathfrak{g}}^1(M)$ equivalent to Poisson str + \mathcal{J}

"Proof"

$$P \in T_{\mathfrak{g}}^1(M) \quad -\lambda + \frac{1}{2} [P, P]_{\mathfrak{g}} = 0$$

$$P = \pi + \mathcal{J} \in (T_{\text{poly}}^1(M))^{\mathfrak{g}} \oplus (\mathfrak{g}^* \otimes T_{\text{poly}}^{-1}(M))^{\mathfrak{g}}$$

$$\frac{1}{2} [P, P]_{\mathfrak{g}}(\xi) = \frac{1}{2} [\pi, \pi] + [\pi, \mathcal{J}_{\xi}] + [\mathcal{J}_{\xi}, \mathcal{J}_{\xi}] = \lambda(\xi)$$

$\begin{matrix} \text{"} & \text{"} & \text{"} \\ 0 & \{ \cdot, \mathcal{J}_{\xi} \} & 0 \end{matrix}$

□

2. Taylor expansion

$$M = C \times \mathfrak{g}^*$$

$$T_{\text{poly}}^k(C \times \mathfrak{g}^*) \cong \bigoplus_{i+j=k} C^\infty(C \times \mathfrak{g}^*) \otimes (\wedge^i \mathfrak{g}^* \otimes T_{\text{poly}}^j(C))$$

First we define

$$T_{\mathfrak{g}^*} : C^\infty(C \times \mathfrak{g}^*)^G \longrightarrow \prod_i (S^i \mathfrak{g} \otimes C^\infty(C))^G \quad \alpha_i \text{ } e^i \text{ word on } \mathfrak{g}^*$$
$$f \longmapsto \sum_{\mathbf{I}} \frac{1}{\mathbf{I}!} e_{\mathbf{I}} \otimes L^* \frac{\partial f}{\partial \alpha_{\mathbf{I}}}$$

- $T_{\mathfrak{g}^*}$ equivariant
- restrict to G -inv
- we can extend

$$T_{\text{poly}}^k(C \times \mathfrak{g}^*) \longrightarrow \prod_i (S^i \mathfrak{g} \otimes \wedge \mathfrak{g}^* \otimes T_{\text{poly}}(C))$$

Def 2.1

$$(S\mathfrak{g}^* \otimes T_{\text{poly}}(\mathbb{C} \times \mathfrak{g}^*))^{\zeta} \xrightarrow{T_{\mathfrak{g}^*}} \underbrace{(S\mathfrak{g}^* \otimes \Pi(S\mathfrak{g} \otimes \wedge \mathfrak{g}^* \otimes T_{\text{poly}}(\mathbb{C})))^{\zeta}}_{T_{\text{Tay}}(\mathbb{C} \times \mathfrak{g}^*)}$$

\rightsquigarrow DGLA

Lemma 2.2 $T_{\mathfrak{g}^*}$ is DGLA morph.

$$\lambda = e^i \otimes (e_i)_M \in T_{\mathfrak{g}^*}^2(M)$$

Corollary 2.3

$$(T_{\mathfrak{g}^*}(M), \lambda, -[J, \cdot], [\cdot, \cdot]) \longrightarrow (T_{\text{Tay}}(\mathbb{C} \times \mathfrak{g}^*), \lambda, -[J, \cdot], [\cdot, \cdot])$$

is a morphism of curved lie algs

π_{KKS} canonical element in T_{Tay}

3. Cartan model

Compute cohomology of T_{Tay}

PROP. 3.1

$$\left(\prod_{i=0}^{\infty} (S^i \mathfrak{g} \otimes T_{poly}(\mathcal{C}))^{\mathfrak{G}}, 0, [\cdot, \cdot] \right) \xrightarrow{\iota} (T_{Tay}(\mathcal{C} \times \mathfrak{g}^*), [-, \cdot], [\cdot, \cdot])$$

↑ !

$$\partial := id \otimes i_s(e^i) \otimes id \otimes (e_i)_c \wedge. \quad \text{DGLA}$$

$$T_{cart}(\mathcal{C}) = \left(\prod_i (S^i \mathfrak{g} \otimes T_{poly}(\mathcal{C}))^{\mathfrak{G}}, 0, [\cdot, \cdot] \right)$$

↑

- Note that we have a canonical DGLA map

$$\rho : (T_{\text{cart}}(\mathcal{G}), \partial, [\cdot, \cdot]) \longrightarrow (T_{\text{poly}}(\mathcal{M}_{\text{red}}), 0, [\cdot, \cdot])$$

PROP. 3.1 ρ is quis.

Find extension of $[-, \cdot]$ to make ι a quis wrt ∂

PROP. 3.2

The map $[\pi_{\text{KKS}}, \cdot]$ is a well-def differential and the inclusion

$$\iota : (T_{\text{cart}}(\mathcal{G}), \partial, [\cdot, \cdot]) \longrightarrow (T_{\text{Tay}}(\mathcal{G} \times \mathfrak{g}^*), [\pi_{\text{KKS}} - \mathcal{J}, \cdot], [\cdot, \cdot])$$

THM. 3.3 \exists \mathcal{L}_{∞} -quis

$$(T_{\text{Tay}}(\mathcal{G} \times \mathfrak{g}^*), [\pi_{\text{KKS}} - \mathcal{J}, \cdot], [\cdot, \cdot]) \longrightarrow (T_{\text{poly}}(\mathcal{M}_{\text{red}}), 0, [\cdot, \cdot])$$

Formal setting :

$$\begin{array}{ccc}
 (T_{\text{Tay}}(C \times_{g^*})[[\hbar]], [\hbar \pi_{\text{KKS}} - J, \cdot]) & \longrightarrow & (T_{\text{poly}}(M_{\text{red}})[[\hbar]], 0, [\cdot, \cdot]) \\
 & \nwarrow \quad \nearrow & \\
 & (T_{\text{cont}}(C)[[\hbar]], \hbar \partial) &
 \end{array}$$

Conclusion :

$$T_g(M)[[\hbar]] \xrightarrow{R} T_{\text{poly}}(M_{\text{red}})[[\hbar]] \quad \text{curved Loo morphism}$$

Remark : reduction procedure of M-W coincides with the one via R [for MC elements of the form

$$\begin{array}{c}
 \hbar \pi \in \hbar T_g(M)[[\hbar]] \\
 \uparrow \\
 T_{\text{poly}}^1(M)
 \end{array}$$