Symplectic Lie Algebroids

GAAG - 2020.

WHAT IS THIS TALK ABOUT?

I. **LIE ALGEBROIDS** ------> ∞-dim Lie algebras of geometric type.

II. **SYMPLECTIC STRUCTURES** -----> phase spaces in classical mechanics.

III. **QUOTIENTS BY SYMMETRIES** ----> reduced phase spaces.

(PH.D. THESIS - DIEGO LOPEZ
MEDELLIN, COLOMBIA)

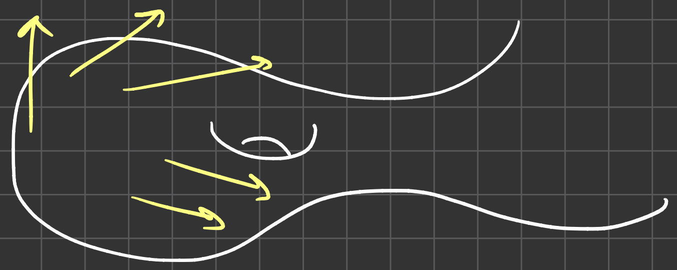
I LIE ALGEBROIDS

① **Lie algebras** : $\underbrace{[\,,\,]}\, : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$

- $\mathbb{R}$ - bilinear & skew - symmetric
- Jacobi identity.

e.g. $\mathfrak{gl}_n(\mathbb{R})$, $\mathfrak{so}(n)$, $\mathfrak{u}(n)$ (FINITE DIMENSIONAL.)

... **Vector fields** :



$\mathfrak{X}(M)$ ⟵ INFINITE DIMENSIONAL !!

$\shortparallel$

$\Gamma(TM)$ ⟶ FINITE DIMENSIONAL OBJECT.

VECTOR BUNDLE

A <u>LIE</u> <u>ALGEBROID</u> is $A \longrightarrow M$

with :

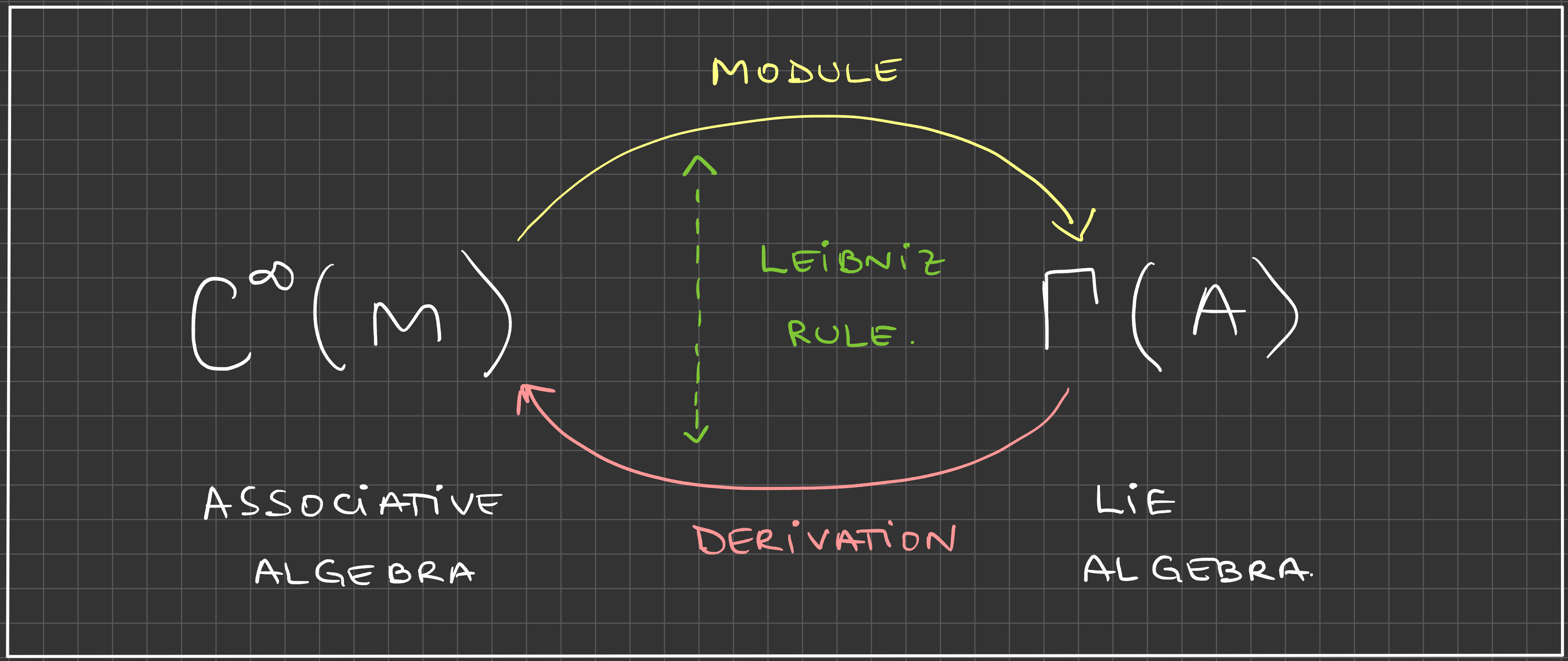
(i) Lie bracket $[\,,\,]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$

(ii) bundle map $\rho_A : A \longrightarrow TM$ (ANCHOR MAP)

satisfying :

$$[a, f \cdot b]_A = f\,[a, b] + \left(\rho_A(a) \cdot f\right) b$$

$$\forall\ a, b \in \Gamma(A)\ ,\ f \in C^\infty(M)$$



Lie-Rinehart pair.

EXAMPLES

(1) Finite dimensional Lie algebra $\rightleftarrows$ LIE ALGEBROID OVER A POINT $A \longrightarrow *$.

(2) Tangent bundle $TM \longrightarrow M$ $\rightleftarrows$

- $A = TM$
- $[\,,\,]_A$ = bracket of vector fields
- $\rho_A = id : TM \longrightarrow TM$

(3) Regular involutive distribution $F \subseteq TM$ $\rightleftarrows$

- $A = F$
- $[\,,\,]_A$ = bracket of vector fields
- $\rho_A = i : F \hookrightarrow TM$

④ Infinitesimal action

$$\mathfrak{g} \longrightarrow \mathfrak{X}(M)$$
$$u \longmapsto u_M$$

$\rightleftarrows$

- $A = M \times \mathfrak{g} \longrightarrow M$
- $\rho_A : M \times \mathfrak{g} \longrightarrow TM$

$$(p, u) \longmapsto u_M(p)$$

- $u, v \in \mathfrak{g} \hookrightarrow \Gamma(A)$

$$[u, v]_A = [u, v]_{\mathfrak{g}}$$

⑤ POISSON STRUCTURES :

$$\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

(i) $\mathbb{R}$-bilinear & skew-symmetric

(ii) $\{f, \cdot\} \in \mathrm{Der}(C^\infty(M))$

(iii) Jacobi identity

EQUIVALENT TO :

(a) bivector $\pi \in \Gamma(\Lambda^2 TM)$

(b) $[\pi, \pi] = 0$

SCHOUTEN BRACKET

Associated Lie Algebroid

$$(M, \pi) \rightsquigarrow \quad \bullet\ A = T^*M \longrightarrow M$$

cotangent bundle

$$\bullet\ \rho_A = T^*M \xrightarrow{\pi^\#} TM$$

$$\bullet\ [\,,\,]_A \text{ on } \Omega^1(M)$$

$$[df, dg]_\pi = d\{f, g\}_\pi$$

$\pi \in \Gamma(\Lambda^2 TM)$

$$T^*M \xrightarrow{\pi^\#} TM$$

$$\alpha \longmapsto \pi(\alpha, \cdot)$$

zero if $\{ , \} \equiv 0$

constant rank

π REGULAR POISSON

isomorphism:

$$TM \xrightarrow{\omega^\#} T^*M$$

$$\underbrace{\begin{array}{l} \omega \in \Omega^2(M) \quad \text{2-form} \\ d\omega = 0 \end{array}}$$

$\omega \in \Omega^2(M)$ SYMPLECTIC FORM.

⑥ b-TANGENT BUNDLES

[MELROSE : differential operators on manifolds with boundary.

NEST-TSYGAN : formal deformations of symplectic manifolds with boundary.]

Def: A **b-MANIFOLD** is a pair (M, Z) where M is a manifold and $Z \subset M$ is a codimension 1 submanifold.

On an adapted local chart $(x_1, \dots, x_n)$

$$Z = \{ x_1 = 0 \}.$$

- Look at ${}^b\mathfrak{X}(M)$ set of vector fields on M tangent to Z.
- Locally free $C^\infty(M)$ – module generated by:

$$x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$$

- it gives ${}^bTM \longrightarrow M$ vector bundle.

b– TANGENT BUNDLE

${}^bTM \longrightarrow M$ is a Lie algebroid !!

- BRACKET: ${}^b\mathfrak{X}(M) \subseteq \mathfrak{X}(M)$

- ANCHOR: ${}^bTM \xrightarrow{\rho} TM$

  induced by the inclusion

  $${}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$$

Lie algebroids vs DGA's

$$(A, [\,,\,]_A, \rho_A) \rightsquigarrow \Omega^p(A) = \Gamma(\Lambda^p A^*)$$

$$\Omega^\bullet(A) = \bigoplus_p \Omega^p(A)$$

$$\Omega^p(A) \xrightarrow{d_A} \Omega^{p+1}(A)$$

$$\omega \longmapsto d_A \omega$$

$$(d_A\omega)(a_1, \dots, a_{p+1}) = \sum_i (-1)^{i+1} \mathcal{L}_{\rho_A(a_i)} \omega(a_1, \dots, \hat{a}_i, \dots, a_{p+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j]_A, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{p+1})$$

PROPERTIES :

(i) $$d_A(\omega \wedge \eta) = (d_A \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge d_A \eta$$

(ii) $$d_A^2 = 0.$$

i.e. $(\Omega^{\bullet}(A), d_A)$ DIFFERENTIAL GRADED ALGEBRA.

CONSEQUENCES:

(i) talk about Lie algebroid **COHOMOLOGY**!!

(ii) natural notion of **MORPHISMS**!!

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d] & B \arrow[d] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd} \rightsquigarrow \quad \Omega^\bullet(B) \xrightarrow{\Phi^*} \Omega^\bullet(A)$$

commutes with the differentials.

II SYMPLECTIC STRUCTURES

We have seen: symplectic 2-form on M is non-degenerate and closed.

Def: A SYMPLECTIC LIE ALGEBROID is a Lie algebroid $A \longrightarrow M$ with $\omega \in \Omega^2(A) = \Gamma(\Lambda^2 A^*)$ non degenerate and closed.

$$\left[\text{i.e.} \quad \begin{array}{l} A \xrightarrow{\omega^\#} A^* \text{ isomorphism} \\ d_A \omega = 0 \end{array} \right]$$

EXAMPLES

1. Symplectic Lie algebras $\dashrightarrow$ infinitesimal data (G, ω) symp. Lie group with ω left-invariant.

   ... affine structures (Medina)

2. Tangent bundle of symplectic manifolds

   - $\omega \in \Omega^2(M) = \Gamma(\Lambda^2 T^*M)$
   - $d_A =$ de Rham.

3. Regular Poisson structure $\pi \in \Gamma(\Lambda^2 TM)$ $\dashrightarrow$
   - $A = \operatorname{im} \pi^\# \subseteq TM$
   - $\omega(\pi^\#(\alpha), \pi^\#(\beta)) = \pi(\alpha, \beta)$

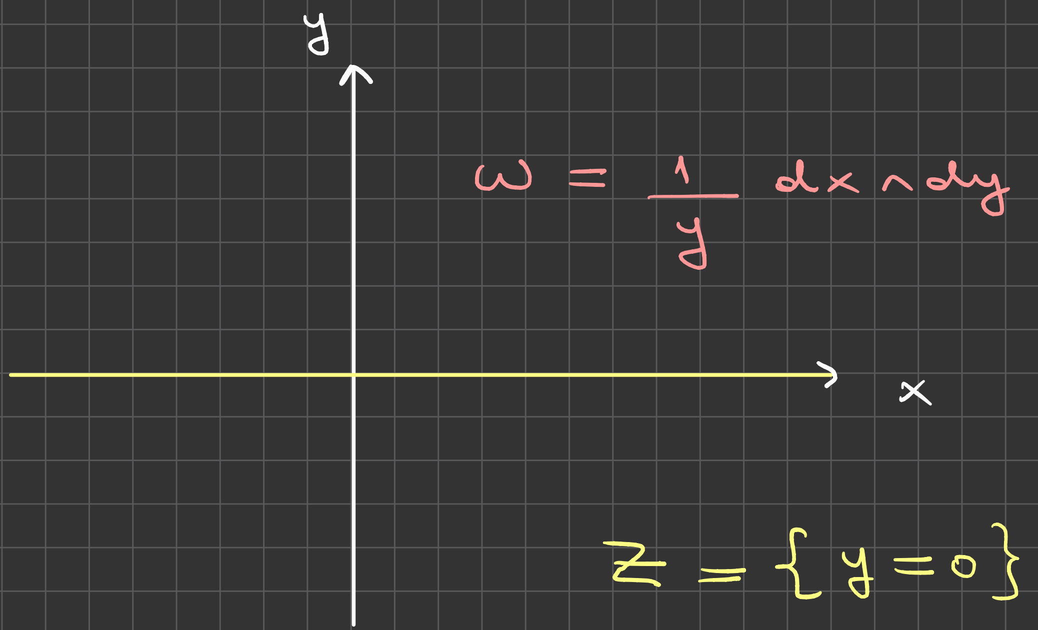
④ Log-symplectic structures.

EXAMPLE: $\mathfrak{g}$ Lie algebra dimension 2.

$$[e_1, e_2] = e_2$$

Poisson structure on $\mathfrak{g}^*$ given by:

$$\pi = y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$



- π is symplectic on an open dense
- explodes on Z but controlled.

Def: A Poisson manifold (M^{2n}, π) is **LOG-SYMPLECTIC** if $\Lambda^n \pi : M \longrightarrow \Lambda^{2n} TM$ vanishes transversally.

$$Z := \{ p \in M \, ; \, (\Lambda^n \pi)(p) = 0 \}$$ ZERO LOCUS

is a codimension 1 submanifold of M.

... look at the b-tangent bundle of (M, Z).

$\rightsquigarrow$ b-cotangent bundle ${}^bT^*M$

$$
\begin{tikzcd}
{}^bT^*M \arrow[r, dashed, "{}^b\pi^\#"'] & {}^bTM \arrow[l, dashed, bend right, "-{}^b\omega^\#"'] \arrow[d, "\rho"] \\
T^*M \arrow[u, "\rho^*"] \arrow[r, "\pi^\#"'] & TM
\end{tikzcd}
$$

$${}^b\omega \in \Gamma\left(\Lambda^2\, {}^bT^*M\right)$$

$\left({}^bTM, -{}^b\omega\right)$ is a symplectic Lie algebroid.

Conclusion: See certain Poisson structure as if they were symplectic!!

OTHER EXAMPLES :

- Ralph Klaasse (Ph. D. Thesis Utrecht 2017.)
- Liu – Sheng – Bai (Dirac structures)
- de León – Marrero – Martínez (Geometric Mechanics)
- Liu – Sheng – Bai – Chen (Left-symmetric algebroids)
- Lyakhovich – Sharapov (BRST quantization)

III Reduction

Example: • $(\mathfrak{g}, \omega)$ symplectic Lie algebra

• $\mathfrak{h} \subseteq \mathfrak{g}$ Lie subalgebra

• $\omega_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{R}$ restriction (NOT NEC. SYMPLECTIC)

• Assume: $\mathfrak{k} := \ker \omega_{\mathfrak{h}} \subseteq \mathfrak{h}$ is an ideal.

so: $\tilde{\mathfrak{h}} := \mathfrak{h}/\mathfrak{k}$ Lie algebra; $\mathfrak{h} \xrightarrow{q} \tilde{\mathfrak{h}}$ surjective Lie algebra map

CONSEQUENCE : $\tilde{\mathfrak{h}}$ inherits a unique $\omega_{\tilde{\mathfrak{h}}} \in \Lambda^2(\tilde{\mathfrak{h}})^*$ symplectic structure characterized by :

$$\underbrace{q^* \omega_{\tilde{\mathfrak{h}}} = \omega_{\mathfrak{h}}.}_{\text{Dardié – Medina "symplectic double extensions of Lie algebras".}}$$

EXAMPLE (Presymplectic reduction).

$\omega \in \Omega^2(M)$ closed with constant rank.

i.e. $F := \ker \omega \subseteq TM$ $\underbrace{\text{regular distribution}}_{\text{INVOLUTIVE !!}}$

Assume: F gives a simple foliation $\mathcal{F}$.

i.e. $\tilde{M} := M/\mathcal{F}$ mfd. & $M \xrightarrow{q} \tilde{M}$ surj. submersion.

$\implies$ $\tilde{M}$ inherits unique symplectic structure $\tilde{\omega}$

characterized by:

$$q^* \tilde{\omega} = \omega.$$

Question (1) $F \subseteq TM$ ideal ?

Question (2) : Notion of ideal in a Lie algebroid ?

Question (3) : Quotients of Lie algebroids ?

Mackenzie – Higgins:

- $K \subseteq A$ Lie subalgebroid
- $\dfrac{A}{K} \in \mathrm{Rep}\left(M \times_{\tilde{M}} M \rightrightarrows M\right)$
- ideal conditions

Fibers quotient

action quotient

$$\begin{tikzcd}
 & \frac{A}{K} \arrow[dr, dashed] & \\
A \arrow[ur, dashed] \arrow[rr, dashed] \arrow[d] & & \tilde{A} \arrow[d] \\
M \arrow[rr, "q"'] & & \tilde{M}
\end{tikzcd}$$

FIBRATION

LIE ALGEBROID

SURJECTIVE

SUBMERSION

- Reduction of symplectic algebroids by ideal systems.

  UNIFIES:
  - Lie algebras
  - Marsden–Weinstein
  - Marrero
  - Log-symplectic reduction. (e.g. Geudens Zambon)

- Reduction of left-symmetric algebroids.
- Reduction of symplectic connections.
- Symplectic double extensions in the algebroid setting.

OTHER QUESTIONS:

① More flexible connections (up to homotopy)

② Log-symplectic case:

- $\pi \in \mathfrak{X}^2(M)$ lifts to a symplectic algebroid structure.

(Gualtieri – Li : symplectic groupoids)

- $(A, \omega) \longrightarrow \underbrace{M}_{\text{Poisson !!}}$ $\rightsquigarrow$ symplectic groupoids?

OBRIGADO !!