Equivariant cohomology for smooth stacks and spectral sequences

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I. STACKS AND GROUP ACTIONS

Quotient Problem for smooth manifolds.

- X smooth manifold, G Lie group with a smooth **free** (and proper) action $\rho : G \times X \to X$. Then we have:
  (i) The quotient X/G exists as a smooth manifold and the quotient morphism $\tau : X \to X/G$ is a submersion and principal G-bundle.

  (ii) For any smooth manifold T and map $f : T \to X/G$ have pullback diagram

$$\begin{array}{ccc} E & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ T & \xrightarrow{f} & X/G \end{array}$$

  So, f defines a submersion and principal G-bundle $\pi : E \to T$ plus G-equivariant morphism $\mu : E \to X$.

- Orbit types of G form stratification of X and give means to study the geometry of X.

- If G is compact, then action is always proper.

- **Question.** What happens if the action is **not** free?

I. STACKS AND GROUP ACTIONS

Let Diff be the (big) site of local diffeomorphisms on the category of smooth manifolds and smooth maps.

Definition 1. A **stack** is a 'sheaf of groupoids' over the site Diff i. e., a pseudofunctor

$$\mathcal{M} : \mathrm{Diff}^{op} \to \mathbf{Grpds}$$

satisfying appropriate glueing axioms for objects, morphisms and coverings with respect to the site Diff. A **morphism of stacks** $F : \mathcal{M} \to \mathscr{N}$ is given by functors for any object $T \in \mathrm{Diff}$

$$F_T^* : \mathcal{M}(T) \to \mathscr{N}(T)$$

and natural transformations for any morphism $f : T' \to T$

$$F_f^* : f^* \circ F_S^* \xrightarrow{\cong} F_{S'}^* \circ f^*.$$

Remark. Stacks over the site Diff form a 2-category **St**

I. STACKS AND GROUP ACTIONS

Example 1. To any smooth manifold $X \in \text{Diff}$ can associate a pseudofunctor, the ***stack*** $\underline{X}$ ***represented by*** X given by

$$\underline{X} = \text{Map}(-, X) : \text{Diff}^{op} \to \textbf{Grpds}$$

which assigns to any smooth manifold $T \in \text{Diff}$ the set of all smooth maps $\text{Map}(T, X)$ between T and X.

Example 2. Let G be a Lie group. Consider the pseudofunctor, the ***classifying stack*** $\mathcal{B}G$ ***of*** G given by

$$\mathcal{B}G : \text{Diff}^{op} \to \textbf{Grpds}$$

which assigns to any smooth manifold $T \in \text{Diff}$ the groupoid $\mathcal{B}G(T)$ of principal G-bundles or G-torsors over T.

Example 3. Let G be a Lie group and $X \in \text{Diff}$ a smooth manifold with action $\rho : G \times X \to X$. Consider the pseudofunctor, the ***quotient stack*** $[X/G]$ given by

$$[X/G] : \text{Diff}^{op} \to \textbf{Grpds}$$

which assigns to any smooth manifold $T \in \text{Diff}$ the groupoid

$$[X/G](T) = \left\langle (E \xrightarrow{\pi} T, E \xrightarrow{\mu} X) : \pi \text{ principal } G\text{-bundle}, \mu \text{ G-equivariant} \right\rangle.$$

Furthermore have

- **(1-morphism)** functors induced by pullbacks of principal G-bundles bundles, i. e. $(f : T' \to T) \mapsto (f^* : [X/G](T) \to [X/G](T'))$
- **(2-morphisms)** natural isomorphism between pullback functors i. e. $(T'' \xrightarrow{g} T' \xrightarrow{f} T) \mapsto (\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*)$

If $X = *$ i. e., a point with trivial G-action, then $[*/G] = \mathcal{B}G$.

I. STACKS AND GROUP ACTIONS

- (**Every sheaf is a stack**) Any sheaf $\mathcal{F} : \mathrm{Diff}^{op} \to \mathrm{Sets}$ is a stack by considering the sets $\mathcal{F}(T)$ as groupoids.

- **(Representable morphisms of stacks)** A morphism of stacks $\mathcal{M} \to \mathscr{N}$ is **representable** if for any morphism $Y \to \mathscr{N}$ the fibre product $\mathcal{M} \times_{\mathscr{N}} Y$ is a stack isomorphic to a smooth manifold X, i. e. to the stack $\underline{X} = Hom_{\mathrm{Diff}}(-, X)$, i.e. the fibre product is represented by a manifold

$$\begin{array}{ccc} \underline{X} \cong \mathcal{M} \times_{\mathscr{N}} Y & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathscr{N} \end{array}$$

- **(2-Yoneda Lemma)** [**Giraud, Hakim**]
Let $\mathcal{M}$ be a stack over Diff and T a smooth manifold. There is an equivalence of categories

$$\theta : Hom_{Stacks}(T, \mathcal{M}) \xrightarrow{\simeq} \mathcal{M}(T),\ (f : T \to \mathcal{M}) \mapsto f(id_T)$$

I. STACKS AND GROUP ACTIONS

Definition 2. A stack $\mathcal{M} : \mathrm{Diff}^{op} \to \mathbf{Grpds}$ is called a **differentiable stack** if there exists a smooth manifold X and a morphism of stacks $p : X \to \mathcal{M}$ such that:

- For any morphism $Y \to \mathcal{M}$ the fibre product $\mathcal{M} \times_{\mathscr{N}} Y$ is a stack isomorphic to a smooth manifold.
- p is a surjective submersion i.e., for any morphism $Y \to \mathcal{M}$, the projection $\mathcal{M} \times_{\mathscr{N}} Y \to Y$ is a surjective submersion.

The morphism $p : X \to \mathcal{M}$ is called an **atlas** or **presentation** for $\mathcal{M}$.

Slogan. We get a 'space' when adding geometry to a set and get a stack when adding geometry to a groupoid. Points in a stack come equipped with a bunch of relations, telling us which points are isomorphic to each other.
A differentiable stack in addition has an atlas given by a 'space' covering the stack, so we can study the geometry of differentiable stacks using the geometry of spaces.

I. STACKS AND GROUP ACTIONS

Theorem 1. $[X/G]$ *is a differentiable stack.*

Proof (Sketch). **Construction of atlas p.** Trivial G-bundle $G \times X \downarrow X$ with action $\rho : G \times X \to X$ gives object in groupoid $[X/G](X)$, i. e. defines a morphism of stacks $p : X \to [X/G]$.

Properties of p. For any smooth manifold T and any morphism $f : T \to [X/G]$, let $\pi : E \downarrow T$ be the corresponding principal G-bundle with G-equivariant morphism $\mu : E \to X$, then $T \times_{[X/G]} X \cong E$.

$$\begin{array}{ccc} E & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & [X/G] \end{array}$$

p is surjective submersion because π is for every f. $\square$

I. STACKS AND GROUP ACTIONS

Quotient Problem for differentiable stacks.

Group actions on stacks
Algebraic stacks: Kontsevich (1995) Laumon-Moret-Baily (2000), Bertin-Mézard (2000), Abramovich-Corti-Vistoli (2003) ..
Romagny (2005): Constructions of quotient and fixed point stacks
Topological stacks: Noohi (2007), Ginot-Noohi (2012) ...

Definition 3. Let G be a Lie group and $\mathcal{M}$ a differentiable stack with atlas $X \to \mathcal{M}$. A G**-action on** $\mathcal{M}$ is a morphism of stacks $\mu : G \times \mathcal{M} \to \mathcal{M}$ together with 2-morphisms α and β, such that for each $T \in$ Diff, the following diagrams

$$\begin{array}{ccc} G \times G \times \mathcal{M}(T) & \xrightarrow{m \times id_{\mathcal{M}(T)}} & G \times \mathcal{M}(T) \\ {\scriptstyle id_G \times \mu_T} \downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow {\scriptstyle \mu_T} \\ G \times \mathcal{M}(T) & \xrightarrow{\mu_T} & \mathcal{M}(T) \end{array}$$

and

$$\begin{array}{ccc} G \times \mathcal{M}(T) & \xrightarrow{\mu_T} & \mathcal{M}(T) \\ {\scriptstyle e \times id_{\mathcal{M}(T)}} \uparrow & \overset{\beta}{\Leftarrow} \quad \nearrow {\scriptstyle id_{\mathcal{M}(T)}} & \\ \mathcal{M}(T) & & \end{array}$$

are 2-commutative, that is, for every $T \in$ Diff the following holds, where the dot $\cdot$ denotes the action μ:

(1) $(g \cdot \alpha^x_{h,k})\alpha^x_{g,hk} = \alpha^{k \cdot x}_{g,h}\alpha^x_{gh,k}$, for all $g, h, k \in G$ and $x \in \mathcal{M}(T)$.
(2) $(g \cdot \beta^x)\alpha^x_{g,e} = 1_{g \cdot x} = \beta^{g \cdot x}\alpha^x_{e,g}$ for every $g \in G$, $x \in \mathcal{M}(T)$ and e the identity in G.

And where $\alpha^x_{g,h} : g \cdot (h \cdot x) \to (gh) \cdot x$ and $\beta^x : x \to e \cdot x$ in $\mathcal{M}(T)$.

Definition 4. The 4-tuple $(\mathcal{M}, \mu, \alpha, \beta)$ is called a G-*stack*, where μ is the G-action on $\mathcal{M}$.

Definition 5. A 1-**morphism of G-stacks** between $(\mathcal{M}, \mu, \alpha, \beta)$ and $(\mathcal{N}, \nu, \gamma, \delta)$ is a morphism of stacks $F : \mathcal{M} \to \mathcal{N}$ together with a 2-morphism σ and the following 2-commutative diagram

$$\begin{array}{ccc} G \times \mathcal{M} & \xrightarrow{\ \mu\ } & \mathcal{M} \\ {\scriptstyle id_G \times f}\big\downarrow & \overset{\sigma}{\swarrow} & \big\downarrow{\scriptstyle f} \\ G \times \mathcal{N} & \xrightarrow{\ \nu\ } & \mathcal{N} \end{array}$$

such that, for every $T \in \text{Diff}$

(1) $\sigma_g^{h\cdot x}(g \cdot \sigma_h^x)\gamma_{g,h}^{F(x)} = F(\alpha_{g,h}^x)\sigma_{gh}^x$, for every $g, h \in G$ and $x \in \mathcal{M}(T)$.

(2) $F(\beta^x)\sigma_e^x = \delta^{F(x)}$, for every object $x \in \mathcal{M}(T)$ and e the identity element of G.

where $\sigma_g^x : F(g \cdot x) \to g \cdot F(x)$ in $\mathcal{N}(T)$.

Definition 6. A **2-morphism of G-stacks** between 1-morphism of G-stacks, (F, σ) and (F', σ'), is a 2-morphism of stacks $\phi : F \Rightarrow F'$ such that

(3) $(\sigma_g^x)(g \cdot \phi_x) = (\phi_{g\cdot x})(\sigma_g'^x)$ for every $g \in G$ and $x \in \mathcal{M}(T)$.

Here $\phi_x : F(x) \to F'(x)$ denotes the 2-morphism ϕ when applied to $x \in \mathcal{M}(T)$.

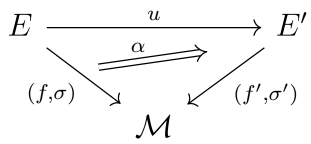
Remark. G-stacks over Diff form a 2-category G–**St**.

Remark. A G-action of a Lie group G on a smooth manifold M coincides with a stacky G-action where the diagrams are now strictly commutative instead. Similarly, the notion of a G-equivariant smooth maps in Diff gives a morphism of G-stacks.

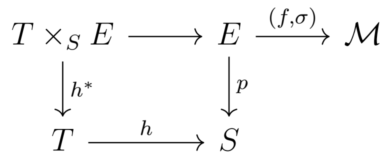
Definition 7 (Quotient stack)**.** Let G be a Lie group acting on a differentiable stack $\mathcal{M}$. Consider the pseudo-functor

$$\mathcal{M}/G : \mathrm{Diff}^{op} \to \mathbf{Grpds}$$

such that for each $T \in \mathrm{Diff}$, an element in $\mathcal{M}/G(T)$ is a triple $t = (p, f, \sigma)$ such that $p : E \to T$ is a principal G-bundle and $(f, \sigma) : E \to \mathcal{M}$ is an equivariant morphism. The arrows in $\mathcal{M}/G(T)$ are pairs (u, α) with a G-morphism $u : E \to E'$ and a 2-commutative diagram of G-stacks



If there is a smooth map $T \xrightarrow{h} S$, then there exists a morphism $\mathcal{M}/G(S) \to \mathcal{M}/G(T)$ given by the pullback diagram



where $\mathcal{M}/G(h) = h^*$.

Proposition 1. *Let G be a Lie group with an action on a differentiable stack $\mathcal{M}$. The pseudofunctor $\mathcal{M}/G$ is a stack.*

Proof. Since it is possible to glue principal G-bundles, the gluing conditions in the definition of a stack hold. Therefore the quotient $\mathcal{M}/G$ is indeed a stack. $\square$

Example 4. Let X be a smooth manifold with an action by a Lie group G. Then we recover the usual quotient stack $[X/G]$ defined for each $T \in \mathrm{Diff}$ via the groupoid of sections as

$$[X/G](T) = \left\langle (E \xrightarrow{\pi} T, E \xrightarrow{\mu} X) : \pi \text{ is a principal } G\text{-bundle}, \mu \text{ is an equivariant map} \right\rangle.$$

Definition 8 (Differentiable G-stack)**.** . Let G be a Lie group. A G-stack $\mathcal{M}$ is called a **differentiable G-stack** if there is a smooth manifold X with a smooth action $\sigma : G \times X \to X$ and a 1-morphism of G-stacks $p : X \to \mathcal{M}$ such that:

(1) p is representable.
(2) p is a submersion.

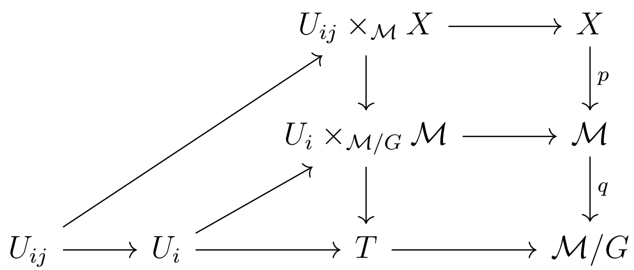
The morphism $p : X \to \mathcal{M}$ is called a **G-atlas for $\mathcal{M}$**.

Remark 1. We obtain a 2-category of differentiable G-stacks denoted by G-Diff**St**.

Proposition 2. *Let $\mathcal{M}$ be a differentiable G-stack with G-atlas given by $X \xrightarrow{p} \mathcal{M}$. If σ is the smooth action of G on X, this action induces a simplicial smooth action $\sigma_\bullet$ on the nerve of the associated Lie groupoid $(X \times_{\mathcal{M}} X \rightrightarrows X)$.*

Proposition 3. *Let $\mathcal{M}$ be a differentiable G-stack and let $X \xrightarrow{p} \mathcal{M}$ be a G-atlas. Then $\mathcal{M}/G$ is a differentiable stack and the composition $X \to \mathcal{M} \xrightarrow{q} \mathcal{M}/G$ is an atlas.*

Proof. We observe that the morphisms p and q have local sections, so it remains to check that $q \circ p$ is representable. We consider the coverings $\{U_i \to T\}$ and $\{U_{ij} \to U_i\}$ such that the first one is a local section for q and the second one a local section for p. Hence we get the following commutative diagram



Thus if we glue every $U_{ij} \times_{\mathcal{M}} X$ using this local section, we get a smooth manifold and as the diagram commutes we see that $T \times_{\mathcal{M}/G} X$ is also a smooth manifold and so the quotient stack $\mathcal{M}/G$ is a differentiable stack. □

Remark. The morphism $q : \mathcal{M} \to \mathcal{M}/G$ is a principal G-bundle, so there exist an associated classifying morphism $u : \mathcal{M}/G \to \mathcal{B}G$ to the classifying stack $\mathcal{B}G$ and we obtain the following 2-cartesian square

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & * \\ \downarrow{\scriptstyle q} & & \downarrow \\ \mathcal{M}/G & \xrightarrow{u} & \mathcal{B}G \end{array}$$

The canonical morphism of differentiable stacks $q : \mathcal{M} \to \mathcal{M}/G$ is actually the universal principal G-bundle over $\mathcal{M}/G$.

Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$. Consider the nerve $X_\bullet$ of its associated Lie groupoid $(X \times_{\mathcal{M}} X \rightrightarrows X)$ and the (simplicial) de Rham complex

$$(\Omega^*_{dR}(X_\bullet), D = d_{dR} + \partial)\,,$$

where ∂ is the differential operator defined via pullbacks of face maps in the simplicial structure of the nerve of $(X \times_{\mathcal{M}} X \rightrightarrows X)$ and d_{dR} is the exterior derivative of the de Rham complex for smooth manifolds and

$$\Omega^n_{dR}(X_\bullet) = \bigoplus_{p+q=n} \Omega^q_{dR}(X_p)$$

with $\Omega^q_{dR}(X_p)$ the sheaf of differential q-forms on the smooth manifold X_p. Let $H^*_{dR}(X_\bullet)$ be the cohomology of $(\Omega^*_{dR}(X_\bullet), D)$. This cohomology is invariant under Morita equivalence of Lie groupoids. Thus, gives well-defined algebraic invariant for differentiable stacks.

Definition 9. Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$. The **de Rham cohomology** of $\mathcal{M}$ is defined as

$$H^*_{dR}(\mathcal{M}) := H^*_{dR}(X_\bullet).$$

The de Rham cohomology of $X_\bullet$ is isomorphic to singular cohomology of its **fat geometric realisation** i.e., of quotient space

$$\|X_\bullet\| = \|p \mapsto X_p\| = \bigcup_{p\in\mathbb{N}} \Delta^p \times X_p / \sim$$

with identifications $(\partial^i t, x) \sim (t, \partial_i x)$ for any $x \in X_p$, $t \in \Delta^{p-1}, i, j = 0, \ldots, n$ and p.

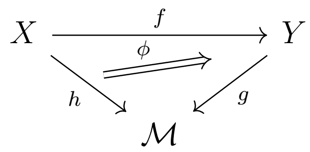
Theorem 2 (de Rham theorem, Behrend 2004)**.** *For a given differentiable stack $\mathcal{M}$, there is an isomorphism*

$$H^*_{dR}(\mathcal{M}) \cong H^*_{sing}(\mathcal{M}, \mathbb{R}).$$

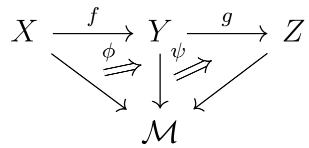
Definition 10. Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$. The *homotopy type* of the differentiable stack $\mathcal{M}$ is given by the homotopy type of the fat geometric realisation $||X_\bullet||$ of the nerve $X_\bullet$ of the associated Lie groupoid $(X \times_{\mathcal{M}} X \rightrightarrows X)$.

II. EQUIVARIANT COHOMOLOGY OF STACKS

Definition 11. A **sheaf $\mathfrak{F}$ (of abelian groups) on a differentiable stack $\mathcal{M}$** is collection of sheaves $\mathfrak{F}_{X\to\mathcal{M}}$ (of abelian groups) for any morphism $X \to \mathcal{M}$, where X is a smooth manifold such that for every triangle



there is sheaf morphism $\Phi_{\phi,f} : f^*\mathfrak{F}_{Y\to\mathcal{M}} \to \mathfrak{F}_{X\to\mathcal{M}}$ such that for



have $\Phi_{\phi,f} \circ f^*\Phi_{\psi,g} = \Phi_{\phi\circ f^*\psi, f\circ g}$. $\mathfrak{F}$ is called **cartesian** if the $\Phi_{\phi,f}$ are isomorphisms.

Let $\mathfrak{F}$ be a sheaf of abelian groups on a differentiable stack $\mathcal{M}$. Consider injective resolution $0 \to \mathfrak{F} \to K^q$ and double complex of global sections

$$N^{\bullet,\bullet} = \Gamma(X_\bullet, K^\bullet)$$

with differentials given by simplicial differential ∂ and resolution differential d_K. This cohomology is the **sheaf cohomology** of $\mathcal{M}$ and will be denoted by $H^*(\mathcal{M}, \mathfrak{F})$.

For complex $\mathfrak{F}^\bullet$ of sheaves of abelian groups on $\mathcal{M}$ with atlas $X \to \mathcal{M}$ get **hypercohomology of** $\mathcal{M}$ *with coefficients* $\mathfrak{F}^\bullet$

$$H^*(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m) = H^*(\mathcal{M}, \mathfrak{F}^\bullet) = H^*(X_\bullet, \mathfrak{F}^\bullet).$$

II. EQUIVARIANT COHOMOLOGY OF STACKS

Models for equivariant cohomology: Borel, Cartan, Getzler

Let G be a Lie group, $\mathcal{M}$ a differentiable G-stack with G-atlas $X \xrightarrow{p} M$. We denote the action on $\mathcal{M}$ by G with $\mu : G \times \mathcal{M} \to \mathcal{M}$ and the action on X by G with $\sigma : G \times X \to X$. Get atlas for the quotient stack $X \xrightarrow{p} \mathcal{M} \xrightarrow{q} \mathcal{M}/G$.

Proposition 4. *There is a* 2*-commutative diagram*

$$\begin{array}{ccc} G \times \mathcal{M} & \xrightarrow{\mu} & \mathcal{M} \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle q} \\ \mathcal{M} & \xrightarrow{q} & \mathcal{M}/G \end{array}$$

and the functor $(pr_2, \mu) : G \times \mathcal{M} \to \mathcal{M} \times_{\mathcal{M}/G} \mathcal{M}$ *is an isomorphism of stacks.*

Therefore, we can consider the following 2-commutative diagram, which is a modification of the one above, using the fact that there is also an induced action of the Lie group G on the atlas X of the stack $\mathcal{M}$:

$$\begin{array}{ccccc} E & \longrightarrow & G \times X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle id_G \times p} & & \downarrow{\scriptstyle p} \\ G \times X & \xrightarrow{id_G \times p} & G \times \mathcal{M} & \xrightarrow{\mu} & \mathcal{M} \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & \mathcal{M} & \xrightarrow{q} & \mathcal{M}/G \end{array}$$

From this we can now conclude the following crucial property

Proposition 5. *There is an equivalence of stacks given by*

$$X \times_{\mathcal{M}/G} X \cong (G \times X) \times_{\mathcal{M}} X \cong G \times (X \times_{\mathcal{M}} X)$$

II. EQUIVARIANT COHOMOLOGY OF STACKS

Borel model

Theorem 3. *Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack with G-atlas $X \to \mathcal{M}$ then*

$$H^*(\mathcal{M}/G, \mathbb{R}) \cong H(EG \times_G \|X_\bullet\|, \mathbb{R}),$$

where $||X_\bullet||$ is the fat geometric realisation of the associated simplicial smooth manifold $X_\bullet$.

Example 5. If $\mathcal{M}$ is a smooth manifold X recover equivariant cohomology for smooth G-manifolds. Namely, since

$$(G \times X) \times_X X \cong G \times X$$

with maps of associated Lie groupoid $(G \times X \rightrightarrows X)$ given by action map $\mu : G \times X \to X$ and projection map $pr_2 : G \times X \to X$ coincides with the transformation groupoid and so gives the Borel construction $EG \times_G X$. Therefore de Rham cohomology of $[X/G]$ is given as:

$$H^*_{dR}([X/G]) \cong H^*(EG \times_G X, \mathbb{R}),$$

In the general situation we get the **Borel model** of equivariant cohomology for the G-stack $\mathcal{M}$

Definition 12. (Borel model) Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack with a G-atlas $X \xrightarrow{p} M$. The **equivariant cohomology** $H^*_G(\mathcal{M}, R)$ of $\mathcal{M}$ is given by

$$H^*_G(\mathcal{M}, R) = H^*(\mathcal{M}/G, R),$$

where R is any commutative ring R with unit.

Remak.(Deligne) This definition of equivariant cohomology in fact makes sense for any cartesian sheaf or complex of cartesian sheaves given on the quotient stack $\mathcal{M}/G$.

II. EQUIVARIANT COHOMOLOGY OF STACKS

Cartan model
Let G be a compact Lie group. Consider the simplicial smooth action $\sigma_\bullet$ induced by σ on $X_\bullet$. Have associated complex $(C^\bullet, D-\iota)$ of simplicial equivariant forms

$$C^{2p+m} = \bigoplus_{q+r=m} \left(S^p(\mathfrak{g}^\vee) \otimes \Omega^q_{dR}(X_r)^G\right),$$

where $\mathfrak{g}$ denotes the Lie algebra of G, with D the differential defined for the de Rham complexand ι is the interior multiplication by the fundamental vector field.

The cohomology of this complex $C^\bullet$ is given by

$$H^*_G(X_\bullet) \cong H^*(EG \times_G ||X_\bullet||, \mathbb{R})$$

And we get the **Cartan model** of equivariant cohomology for the G-stack $\mathcal{M}$

$$H^*_G(\mathcal{M}, \mathbb{R}) = H^*(\mathcal{M}/G, \mathbb{R}) \cong H^*_G(X_\bullet).$$

We have the double complex $(C^{\bullet,\bullet}, D, \iota)$ of invariant forms with

$$C^{p,q} = \left(S^p(\mathfrak{g}^\vee) \otimes \left(\bigoplus_{s+r=q-p} \Omega^s_{dR}(X_r)\right)\right)^G$$

with vertical operator D and horizontal operator ι.

Filtering this double complex $C^{\bullet,\bullet} = \{C^{p,q}\}$ in the standard way gives rise to a spectral sequence

Theorem 4. *The E_1-term of the spectral sequence for the double complex $C^{\bullet,\bullet}$ of invariant forms is given as*

$$E_1^{p,q} = (S^p(\mathfrak{g}^\vee) \otimes H^{q-p}(X_\bullet, \mathbb{R}))^G \Rightarrow H^*_G(\mathcal{M}, \mathbb{R})$$

If G is connected, then $E_1^{p,q} = S^p(\mathfrak{g}^\vee)^G \otimes H^{q-p}(X_\bullet, \mathbb{R})$

II. EQUIVARIANT COHOMOLOGY OF STACKS

Theorem 5. *Let G be compact connected Lie group, K closed subgroup of G. Suppose that the restriction map*

$$S(\mathfrak{g}^\vee)^G \to S(\mathfrak{k}^\vee)^K$$

is an isomorphism. Then the induced map in equivariant cohomology is also an isomorphism

$$H_G(\mathcal{M}, \mathbb{R}) \xrightarrow{\cong} H_K(\mathcal{M}, \mathbb{R}).$$

Theorem 6. *Let G be connected compact Lie group, T maximal torus of G, W the Weyl group and $\mathcal{M}$ be a differentiable G-stack. Then we have*

$$H^*_G(\mathcal{M}, \mathbb{R}) \cong H^*_T(\mathcal{M}, \mathbb{R})^W.$$

Getzler model

Let G be any Lie group. Consider $X_\bullet = \{X_n\}$. Define vector spaces $C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*_{dR}(X_n))$ of smooth maps

$$G^p \to S^*(\mathfrak{g}^\vee) \otimes \Omega^*_{dR}(X_n)$$

from the p-fold cartesian product G^p to $S^*(\mathfrak{g}^\vee) \otimes \Omega^*_{dR}(X_n)$, i.e. polynomial maps from $\mathfrak{g}$ to differential forms on X_n. The complex $C^\bullet$ is then given by

$$\bigoplus_{p+2l+m=s} C^p(G, S^l(\mathfrak{g}^\vee) \otimes \Omega^m_{dR}(X_n))$$

endowed with the differential $\mathcal{D} + \bar{\iota} + (-1)^p(d + \iota)$, where $d + \iota$ is the Cartan differential operator as in the Cartan model, the operator $\mathcal{D}$ is given by

$$\mathcal{D} : C^p(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*_{dR}(X_n)) \to C^{p+1}(G, S^*(\mathfrak{g}^\vee) \otimes \Omega^*_{dR}(X_n))$$

such that

$$(\mathcal{D}f)(g_0, \ldots, g_k \mid Y) := f(g_1, \ldots, g_k \mid Y) + \sum_{i=1}^{k} f(g_0, \ldots, g_{i-1}g_i, \ldots, g_k \mid Y)$$
$$+(-1)^{k+1} f(g_0, \ldots, g_{k-1} \mid \mathrm{Ad}(g_k^{-1})Y)$$

for $g_0, \ldots, g_k \in G$ and $Y \in \mathfrak{g}$, and the operator $\bar{\iota}$ is defined by

$$\bar{\iota} : C^p(G, S^l(\mathfrak{g}^\vee) \otimes \Omega^m_{dR}(X_n)) \to C^{p-1}(G, S^{l+1}(\mathfrak{g}^\vee) \otimes \Omega^m_{dR}(X_n))$$

with

$$(\bar{\iota} f)(g_1, \ldots, g_{p-1} \mid Y) := \sum_{i=0}^{p-1} \frac{d}{dt}\bigg|_{t=0} f(g_1, \ldots, g_i, \exp(tY_i), g_{i+1}, \ldots, g_{p-1} \mid Y)$$

where $Y_i = \mathrm{Ad}(g_{i+1} \cdots g_{p-1})Y$.

Can also consider the simplicial differential ∂_X for the associated simplicial smooth manifold $X_\bullet$. Because ∂_X commutes with operator $\mathcal{D}+\bar{\iota}+(-1)^p(d+\iota)$, get complex $C^\bullet$ with differential $\mathcal{D}+\bar{\iota}+(-1)^p(d+\iota)+(-1)^{p+2l+m}\partial_X$.

$$\bigoplus_{p+2l+m+n=s} C^p(G, S^l(\mathfrak{g}^\vee)\otimes\Omega^m_{dR}(X_n))$$

The cohomology $H^*_G(X_\bullet)$ gives the **Getzler model** for equivariant cohomomology for the G-stack $\mathcal{M}$

Theorem 7. *Let $\mathcal{M}$ be a differentiable G-stack with G-atlas $X\to\mathcal{M}$ for a general Lie group G. Then there is an isomorphism*

$$H^*_G(\mathcal{M},\mathbb{R})\cong H^*_G(X_\bullet).$$

Proof. Use spectral sequence argument. Let us consider the complex $\bigoplus_{p+2l+m+n=s} C^p(G, S^l(\mathfrak{g}^\vee)\otimes\Omega^m_{dR}(X_n))$ as the double complex $E_0^{'\bullet,\bullet}$ with

$$\left(E_0^{'s,n}=\left(\bigoplus_{p+2l+m=s,n} C^p(G, S^l(\mathfrak{g}^\vee)\otimes\Omega^m_{dR}(X_n))\right), \mathcal{D}+\bar{\iota}+(-1)^p(d+\iota), \partial_X\right)$$

In the same way, we consider the triple complex $(\Omega^q_{dR}(G^p\times X_n), d_{dR}, \partial_G, \partial_X)$ as the double complex $E_0^{\bullet,\bullet}$ with

$$\left(E_0^{s,n}=\left(\bigoplus_{p+q=s,n}\Omega^q_{dR}(G^p\times X_n)\right), d_{dR}+(-1)^q\partial_G, \partial_X\right)$$

If we calculate both cohomologies first for the index s, we will get the same cohomology, because the cohomology of $E_0^{s,n}$ is the cohomology of the associated Lie groupoid $(G\times X_n\rightrightarrows X_n)$ for the action of G on the smooth manifold X_n. Therefore $E_1^{'s,n}\cong E_1^{s,n}$, so both spectral sequences are isomorphic in the first page and converge to same cohomology $H^*_G(\mathcal{M},\mathbb{R})$. □

III. SPECTRAL SEQUENCES FOR EQUIVARIANT STACK COHOMOLOGY

Let G be a Lie group, $\mathcal{M}$ a differentiable G-stack and $\mathfrak{F}$ a (cartesian) sheaf on the quotient stack $\mathcal{M}/G$. Consider atlas $X \to \mathcal{M} \to \mathcal{M}/G$, where $X \to \mathcal{M}$ is a G-atlas for $\mathcal{M}$ then we get an induced sheaf $\mathfrak{F}$ on $\mathcal{M}$ and a bisimplicial sheaf $\mathfrak{F}_\bullet$ on the associated bisimplicial smooth manifold $Z_{\bullet,\bullet}$ with $Z_{p,n} = G^p \times X_n$. Get triple complex $K^{\bullet,\bullet,\bullet}$ given by $\Gamma(G^p \times X_n, K^{p,n,q})$ using acyclic resolution of $\mathfrak{F}$

Theorem 8. *There exists a spectral sequence such that*

$$E_1^{r,n} = H^r([X_n/G], \mathfrak{F}) \Rightarrow H_G^{r+n}(\mathcal{M}, \mathfrak{F}).$$

Theorem 9. *If G is a countable discrete group, there exists a spectral sequence such that*

$$E_2^{p,r} = H^p(G, H^r(\mathcal{M}, \mathfrak{F})) \Rightarrow H_G^{p+r}(\mathcal{M}, \mathfrak{F}).$$

Example 6. (Felder-Henriques-Rossi-Zhu)
If $\mathcal{M}$ is just a smooth manifold X with an action by a discrete group G, the above result gives a spectral sequence of the form

$$E_2^{p,n} = H^p(G, H^n(X, \mathfrak{F})) \Rightarrow H^{p+n}([X/G], \mathfrak{F})$$

III. SPECTRAL SEQUENCES FOR EQUIVARIANT STACK COHOMOLOGY

Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack. Furthermore, let $\mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m$ be a complex of (cartesian) sheaves of abelian groups on $\mathcal{M}/G$ with an atlas given by $X \to \mathcal{M} \to \mathcal{M}/G$, where $X \to \mathcal{M}$ is a G-atlas for $\mathcal{M}$. For any r, let $\mathfrak{F}_r$ be the sheaf from the complex on $\mathcal{M}/G$ and $\mathfrak{F}_{r,\bullet}$ be the induced bisimplicial sheaf on the associated bisimplicial smooth manifold $G^\bullet \times X_\bullet$. We can relate the equivariant hypercohomology of the nerve with the equivariant hypercohomology of the stack as follows

Theorem 10. *There exists a spectral sequence such that*

$$E_1^{s,n} = H^s([X_n/G], \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m) \Rightarrow H_G^{s+n}(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m).$$

Theorem 11. *If G is a countable discrete group, there exists a spectral sequence such that*

$$E_2^{p,s} = H^p(G, H^s(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m)) \Rightarrow H_G^{p+s}(\mathcal{M}, \mathfrak{F}_0 \to \mathfrak{F}_1 \cdots \to \mathfrak{F}_m).$$

Example 7. (Felder-Henriques-Rossi-Zhu)
If $\mathcal{M}$ is just smooth manifold X with an action of a discrete group G, we get a spectral sequnce

$$E_2^{p,n} = H^p(G, H^n(X, \mathfrak{F}_0 \to \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_m)) \Rightarrow H^{p+n}([X/G], \mathfrak{F}_0 \to \mathfrak{F}_1 \cdots \to \mathfrak{F}_m)$$

III. SPECTRAL SEQUENCES FOR EQUIVARIANT STACK COHOMOLOGY

Bott spectral sequence
Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack. We get the following generalised version of Bott's spectral sequence, where $H_c^*(-)$ denotes continuous cohomology

Theorem 12. *There exists a spectral sequence such that*

$$E_1^{k,p} = \bigoplus_{q+n=k} H_c^p(G, \bigoplus_{s+t=q} \Omega_{dR}^s(X_n) \otimes S^t(\mathfrak{g}^\vee)) \Rightarrow H_G^{k+p}(\mathcal{M}, \mathbb{R})$$

and where

$$E_2^{k,p} = Tot \bigoplus_{q+n=k} H_c^p(G, \bigoplus_{s+t=q} \Omega_{dR}^s(X_n) \otimes S^t(\mathfrak{g}^\vee)).$$

Example 8. (Bott spectral sequence) If $\mathcal{M}$ is a point $*$ with trivial G-action, the above spectral sequence reduces to the following spectral sequence converging to the cohomology of the classifying space of the Lie group G

$$E_1^{t,p} = H_c^p(G, S^t(\mathfrak{g}^\vee)) \Rightarrow H^{t+p}(BG, \mathbb{R})$$

as the homotopy type of the classifying stack $\mathcal{B}G = [*/G]$ is given by the classifying space BG of G.
If G is compact, Bott's spectral collapses and recovers the classical Borel isomorphism

$$H^*(BG, \mathbb{R}) \cong S^*(\mathfrak{g}^\vee)^G.$$

IV. MORE APPLICATIONS

(work in progress...)

Definition 13. Let $\mathcal{M}$ be a differentiable stack and p a non-negative integer and $\mathbb{T}$ the sheaf of circle valued functions. The **smooth Deligne complex** $\mathcal{F}(p)^\infty_\mathcal{D}$ is the following complex of sheaves on $\mathcal{M}$:

$$\mathcal{F}(p)^\infty_\mathcal{D} : \mathbb{T}_\mathcal{M} \xrightarrow{\frac{1}{2\pi i} d\log} \Omega^1_\mathcal{M} \xrightarrow{d} \Omega^2_\mathcal{M} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p_\mathcal{M}.$$

The **smooth Deligne cohomology** $H^q_\mathcal{D}(\mathcal{M}, \mathcal{F}(p)^\infty)$ of $\mathcal{M}$ with respect to $\mathcal{F}(p)$ is defined to be the hypercohomology group of the smooth Deligne complex

$$H^q_\mathcal{D}(\mathcal{M}, \mathcal{F}(p)^\infty) = \mathbb{H}^q(\mathcal{M}, \mathcal{F}(p)^\infty_\mathcal{D}).$$

Theorem 13. *Let $\mathcal{M}$ be a differentiable stack. Then we have:*

(i) *The isomorphism classes of principal $\mathbb{T}$-bundles over $\mathcal{M}$, i. e. hermitian line bundles over $\mathcal{M}$ are classified by cohomology classes in $H^1(\mathcal{M}, \mathbb{T}_\mathcal{M}) \cong H^2(\mathcal{M}, \mathbb{Z})$.*

(ii) *The isomorphism classes of G-equivariant principal $\mathbb{T}$-bundles over $\mathcal{M}$ are classified by cohomology classes in $H^1_G(\mathcal{M}, \mathbb{T}_\mathcal{M}) \cong H^2_G(\mathcal{M}, \mathbb{Z})$.*

(iii) *The isomorphism classes of gerbes with band $\mathbb{T}$ over $\mathcal{M}$ are classified by cohomology classes in $H^2(\mathcal{M}, \mathbb{T}_\mathcal{M}) \cong H^3(\mathcal{M}, \mathbb{Z})$.*

Theorem 14. *Let $\mathcal{M}$ be a differentiable stack. Then we have:*

(i) *The isomorphism classes of principal $\mathbb{T}$-bundles with connection over $\mathcal{M}$ are classified by cohomology classes in $H^1_\mathcal{D}(\mathcal{M}, \mathcal{F}(1)^\infty)$.*

(ii) *The isomorphism classes of gerbes with band $\mathbb{T}$ with connective structure and curving over $\mathcal{M}$ are classified by cohomology classes in $H^2_\mathcal{D}(\mathcal{M}, \mathcal{F}(2)^\infty)$.*