Lagrangian submanifolds as modules

GAAG VI

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<u>Lagrangian submanifolds</u>

(M^{2n}, ω) symplectic, $L^n \subset M$ is <u>Lagrangian</u> if $\omega|_L \equiv 0$.

<u>Ex</u>: $M^{2n} = T^*Q^n$, $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$ cotangent bundle of smooth Q.

$L = Q$ is a Lagrangian. So is a fiber $T^*_q Q$, for $q \in Q$.

Cotangent bundles are <u>exact</u>:

$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i = d\left(\sum_{i=1}^{n} p_i \, dq_i\right) =: d\lambda$, λ 1-form on T^*Q.

$L \subset T^*Q$ is <u>exact</u> if $\lambda|_L = df$, for some function $f : L \to \mathbb{R}$.

<u>Ex</u>: $Q, T^*_q Q \subset T^*Q$ are exact Lagrangians.

<u>Arnold Conjecture</u>: If Q is smooth and closed (compact, no boundary), then the only closed <u>exact</u> Lagrangian $L \subset T^*Q$ is Q, up to Hamiltonian isotopy (φ_H^1, for φ_H^t flow of X_H^t st $\omega(\cdot, X_H^t) = d_x H(t,x)$ for (Hamiltonian) function $H : \underset{t}{[0,1]} \times \underset{x}{T^*Q} \to \mathbb{R}$)

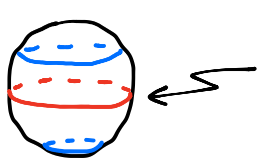
$(\varphi_H^1)^* \omega = \omega$

Arnold's (nearby Lagrangian) conjecture is still open, except for $Q = S^1, S^2, T^2$

<u>General classification question</u>: What are the Lagrangian submanifolds $L \subset (M^{2n}, \omega)$, up to Hamiltonian isotopy?

<u>Today</u>:
* $M^{2n} = T^*S^n$ cotangent bundle of a sphere, <u>$n \geq 2$</u>.
* $L^n \subset M^{2n}$ <u>monotone</u> Lagrangian:

$\boxed{[\omega] = \tau_L \cdot \mu} \in H^2(M, L; \mathbb{R})$, where
- $\tau_L \geq 0$ constant (if $\omega = d\lambda$ and L exact, then $\tau_L = 0$)
- μ is the <u>Maslov class</u>

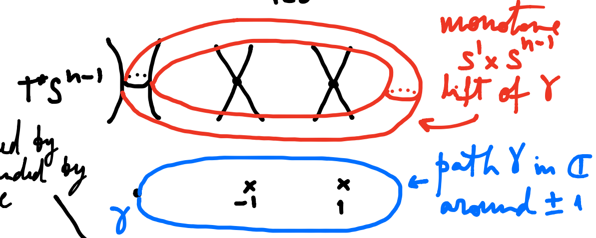
<u>Ex</u>:  the equator is the only parallel circle in (S^2, ω_{std}) that is a monotone Lagrangian.

<u>Note</u>: Monotonicity helps with technical aspects of pseudoholomorphic curves.

Compact Monotone Lagrangians in T^*S^n:

- S^n zero section is exact
- T^*S^n is symplectomorphic to complex affine quadric
  $Q_n = \{\sum_{i=0}^{n} z_i^2 = 1\} \subset \mathbb{C}^{n+1}$ (with restriction of $\omega_{std} = \sum_{i=0}^{n} \frac{i}{2} dz_i \wedge d\bar{z}_i$ on $\mathbb{C}^{n+1}$)

Lefschetz fibration: $\begin{array}{ccc} Q_n ; & (z_0, \dots, z_n) \\ \downarrow & \downarrow \\ \mathbb{C} & z_0 \end{array}$

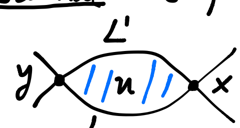


$\rightsquigarrow$ 1-parameter family $\boxed{(S^1 \times S^{n-1})_\tau}$, $\tau > 0$, of monotone Lag'ns.

Thm 1 (Abouzaid-D., '20): $L \subset T^*S^n$ closed monotone (orient. spin) st $HF^*(L,L) \neq 0$. Then,
i) either $HF(L, S^n) \neq 0$ ($\Rightarrow L$ not displaceable from S^n by a Ham. isotopy);
ii) or $HF(L, (S^1 \times S^{n-1})_\tau) \neq 0$ for some $\tau > 0$ ($\Rightarrow$ ~ $(S^1 \times S^{n-1})_\tau$ ~).

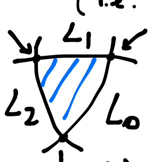
Lagrangian Floer cohomology: Fix a field $\mathbb{F}$ (want: Novikov field over $\mathbb{C}$)
$L, L' \subset (M^{2n}, \omega)$ Lagrangians $\rightsquigarrow$ chain complex $CF^*(L, L')$:

- $CF^*(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{F}\langle x \rangle$ (assume $L \pitchfork L'$ transverse intersection)
- differential: $\partial = \mu^1$ st $\langle \partial x, y \rangle$ counts pseudoholomorphic strips:



$u : \mathbb{R} \times [0,1] \longrightarrow \mathbb{R}$ st $\boxed{\bar{\partial}_J u = 0}$ ← Cauchy-Riemann equation
(J almost complex structure on M, compatible with ω)
figure specifies boundary & asymptotic conditions on u

Properties: - $\partial^2 = 0 \rightsquigarrow HF^*(L, L')$ Lagrangian Floer cohomology;
- $HF^*(L, L') \neq 0 \Rightarrow L$ not displaceable from L' by a Hamiltonian isotopy
  (i.e. $\varphi_H^1(L) \cap L' \neq \emptyset$ for every $H : [0,1] \times M \to \mathbb{R}$);
- Product: count



to define $\mu^2 : HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2)$
μ^2 not associative on CF^*, μ^k are homotopies of homotopies...
- Higher polygons

define A_∞-operations μ^k (with k inputs)

Fukaya categories

$\mathrm{Fuk}(M^{2n}, \omega)$: an A_∞-category (composition associative up to homotopy) st:

- objects : closed, oriented, spin, monotone Lagrangians $L \subset M$
  (+ unitary local system, bounding cochain (i.e. MC elt. in $CF^*(L,L)$), ...)
- morphisms : $\hom(L, L') = CF^*(L, L')$ (Floer cochain complex)
- A_∞-operations : μ^k (satisfy "$\mu \circ \mu = 0$")

M^{2n} open (e.g. T^*Q) $\rightsquigarrow \mathcal{W}\mathrm{Fuk}(M, \omega)$ wrapped Fukaya category

- objects : Lagrangians that are either closed, or "cylindrical at infinity"
  Ex : cotangent fibers $T_q^*Q \subset T^*Q$. We'll only need these non-compact Lags.
- morphisms : wrapped Floer cochains : $CW^*(L, L') = CF^*(L, \varphi_H^1(L'))$, where H is a Hamiltonian growing fast at infinity.
- A_∞-operations : analogues of above

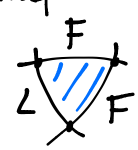
Abouzaid : A cotangent fiber generates $\mathcal{W}\mathrm{Fuk}(T^*Q)$ (in triangulated sense).

Let $\mathcal{A} := CW^*(F, F)$ be the A_∞-algebra of a cotangent fiber $F = T_q^*Q$.

$\Rightarrow$ (Yoneda) functor $\mathcal{Y} : \mathcal{W}\mathrm{Fuk}(T^*Q) \longrightarrow \mathrm{mod}_{\mathcal{A}}$, $L \longmapsto CW^*(F, L)$ is cohomologically fully faithful

Note : • $\mathrm{mod}_{\mathcal{A}}$ is the category of (right) A_∞-modules over $\mathcal{A}$.

• module structure from $\mu^2 : CW^*(F,L) \otimes CW^*(F,F) \longrightarrow CW^*(F,L)$



Assume $\boxed{Q = S^{n \geq 2}}$. Then, $\mathcal{A} = CW^*(F,F) \cong C_{-*}(\Omega_q S^n)$ and $\boxed{A} := H^*(\mathcal{A}) \cong \boxed{\mathbb{F}[x]}$, $\deg(x) = 1-n$

Prop 1 : • $\mathcal{A}$ is a formal A_∞-algebra (quasi-iso to coho $A = H^*\mathcal{A}$)

• every $\mathcal{M} \in \mathrm{mod}_{\mathcal{A}}$ is formal (quasi-iso to A-module $M = H^*\mathcal{M}$)

Cor 1 : The functor $Y : \mathcal{W}\mathrm{Fuk}(T^*S^n) \longrightarrow \mathrm{Mod}_A$, $L \longmapsto HW^*(F, L)$ is cohomologically fully-faithful

Note : objects in Mod_A are A-modules (not A_∞), $\hom_{\mathrm{Mod}_A}(M,N) = \mathrm{Ext}^*(M,N)$.

Upshot : Can $\boxed{\text{replace}}$ the study of $\boxed{\mathcal{W}\mathrm{Fuk}(T^*Q)}$ by the study of $\boxed{\mathrm{Mod}_A}$

Compact objects in Mod_A (recall: $A = \mathbb{F}[x]$)

If $L \subset T^*Q$ is a closed monotone Lagr, then $\dim_{\mathbb{F}} HW^*(F,L) < \infty$.

Def: $Mod_A^{pr} \subset Mod_A$ is the subcategory of fin. dim'l. modules.

Ex: Given $\alpha \in \mathbb{F}$, let S_α be a 1-dim'l $\mathbb{F}$-vector space, with $x : S_\alpha \to S_\alpha$ st $x(v) = \alpha \cdot v$.

Prop 2: If $\mathbb{F}$ is algebraically closed, then the collection $\{S_\alpha\}_{\alpha \in \mathbb{F}}$ generates Mod_A^{pr}. eg: $\mathbb{F}$ = Novikov field over $\mathbb{C}$

Proof: Since $\mathbb{F}$ is alg. closed, any $M \in Mod_A^{pr}$ can be written, as a module over $A = \mathbb{F}[x]$, in Jordan form. All Jordan blocks $\begin{pmatrix} \alpha & 1 & & 0 \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{pmatrix}$ can be generated by the S_α. ∎

Prop 3: The collection $\{Y(S^n)\} \cup \{Y((S^1 \times S^{n-1})_\tau)\}_{\tau > 0} \subset Mod_A^{pr}$ (where S^n is equipped w/ a bounding cochain & $(S^1 \times S^{n-1})_\tau$ w/ a local system) split-generates all the S_α, $\alpha \in \mathbb{F}$.

Proof: Calculation of HF^* and μ^2.

Proof of Thm 1:

Thm 1 (Abouzaid-D., '20): $L \subset T^*S^n$ closed monotone (...) st $HF^*(L,L) \neq 0$. Then,
i) **either** $HF(L, S^n) \neq 0$ ii) **or** $HF(L, (S^1 \times S^{n-1})_\tau) \neq 0$ for some $\tau > 0$.

Know: **Cor 1**: $Y : Fuk(T^*S^n) \longrightarrow Mod_A^{pr}$, $L \longmapsto HW^*(F,L)$ is cohomologically fully-faithful

Prop 2: If $\mathbb{F}$ alg. closed, then the collection $\{S_\alpha\}_{\alpha \in \mathbb{F}}$ generates Mod_A^{pr}.

Prop 3: $\{Y(S^n)\} \cup \{Y((S^1 \times S^{n-1})_\tau)\}_{\tau > 0}$ split-generate all the S_α, $\alpha \in \mathbb{F}$.

Cor 2: The collection $\{S^n\} \cup \{(S^1 \times S^{n-1})_\tau\}_{\tau > 0}$ split-generates $Fuk(T^*S^n)$.

Proof: Prop 3 + Prop 2 + Cor 1. ∎

Proof of Thm 1: $HF^*(L,L) \neq 0 \Rightarrow L$ is a non-trivial object in $Fuk(T^*Q)$. By Cor 2, it can be "split-generated" by S^n or $(S^1 \times S^{n-1})_\tau$. ∎

Note: think of Cor 2 as a (weak) Floer-theoretic classif. of monotone Lagrs in T^*S^n.