Virtual Euler characteristics of moduli spaces of torsion free sheaves on general type surfaces

joint with L. Göttsche and T. Laarakker

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Method: use geometric invariant theory

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Assume: stable=semistable e.g. $\gcd(r, c_1 \cdot H) = 1$
$\Rightarrow M_S^H(r, c_1, c_2)$ projective

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Smooth case: $\sum_{c_2} e(M_S^H(r, c_1, c_2))\, q^{c_2}$ calculated in examples

Göttsche, Klyachko, Göttsche-Huybrechts, Yoshioka, Manschot, Mozgovoy, Weist, K, ...

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Theorem (Göttsche, 1990)

$$\sum_n e(\mathrm{Hilb}^n(S))\, q^{n - \frac{e(S)}{24}} = \eta(q)^{-e(S)}$$

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- higher rank: Manschot

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Virtual class Behrend-Fantechi, Li-Tian; Mochizuki

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Virtual Euler char. CIOCAN-FONTANINE-KAPRANOV,FANTECHI-GÖTTSCHE

$e^{\mathrm{vir}}(M) := \int_{[M]^{\mathrm{vir}}} c_{\mathrm{vdim}(M)}(T_M^{\mathrm{vir}}) \in \mathbb{Z}$

$$\Theta_{A_r^\vee,\ell}(q) := \sum_{v\in\mathbb{Z}^r} e^{2\pi i\langle v,\ell\lambda\rangle^\vee} q^{\frac{1}{2}\langle v,v\rangle^\vee}, \quad \lambda := (1,0,\ldots,0)$$

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Conjecture (Göttsche-K, Göttsche-K-Laarakker)

For $r = 2,3,5$, $e^{\mathrm{vir}}(M_S^H(r,c_1,c_2))$ *equals*

$$\mathrm{Coeff}_{q^{c_2-\frac{r-1}{2r}c_1^2-\frac{r}{2}\chi+\frac{r}{24}K^2}}\left\{ r^{2+K^2-\chi}\left(\frac{1}{\Delta(q^{\frac{1}{r}})^{\frac{1}{2}}}\right)^{\chi}\left(\frac{\Theta_{A_{r-1}^\vee,0}(q)}{\eta(q)^r}\right)^{-K^2}\Psi_{S,r,c_1}(q)\right\}$$

with $\Psi_{S,r,c_1}(q)$ *given below*

$r = 2$: Vafa–Witten (cosmic string)

$$\Psi_{S,2,c_1} = 1 + (-1)^{c_1 K + \chi} \left(\frac{\Theta_{A_1^\vee,0}}{\Theta_{A_1^\vee,1}} \right)^{K^2}$$

$r = 3$: Göttsche–K

$$\Psi_{S,3,c_1} = \left(\frac{\Theta_{A_2^\vee,0}}{\Theta_{A_2^\vee,1}}\right)^{K^2} (X_+^{K^2} + X_-^{K^2}) + (\epsilon_3^{c_1 K} + \epsilon_3^{-c_1 K})(-1)^{\chi} \left(\frac{\Theta_{A_2^\vee,0}}{\Theta_{A_2^\vee,1}}\right)^{K^2}$$

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$$\Psi_{S,3,c_1} = \left(\frac{\Theta_{A_2^\vee,0}}{\Theta_{A_2^\vee,1}}\right)^{K^2} (X_+^{K^2}+X_-^{K^2})+(\epsilon_3^{c_1 K}+\epsilon_3^{-c_1 K})(-1)^\chi \left(\frac{\Theta_{A_2^\vee,0}}{\Theta_{A_2^\vee,1}}\right)^{K^2}$$

where $X_\pm$ are the solutions of

$$X^2 - 4\left(\frac{\Theta_{A_2^\vee,0}}{\Theta_{A_2^\vee,1}}\right)^2 X + 4\frac{\Theta_{A_2^\vee,0}}{\Theta_{A_2^\vee,1}} = 0$$

(corrects Labastida-Lozano)

$r = 5$: Göttsche–K–Laarakker

$$t_\ell := \frac{\Theta_{A_4^\vee,\ell}}{\Theta_{A_4^\vee,0}}, \quad \rho := \frac{1-\phi R}{\phi+R}$$

$$R := \cfrac{q^{\frac{1}{5}}}{1+\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}}, \quad \text{Rogers–Ramanujan}$$

$$\phi := \frac{1+\sqrt{5}}{2}, \quad \text{golden ratio}$$

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$$\beta_1 := \frac{1}{25t_1}\left\{3\rho^{-5}+2-8\rho^5\right\}, \quad \beta_2 := \frac{1}{25t_2}\left\{8\rho^{-5}+2-3\rho^5\right\}$$

$$\alpha+\alpha^{-1} := \frac{6}{25}\left\{8\rho^{-5}-13-8\rho^5\right\} = \frac{240}{j_5(q^{\frac{1}{5}})}+18$$

$$j_5 := \left(\frac{\eta(q)}{\eta(q^5)}\right)^6 \quad \text{Hauptmodul } \Gamma_0(5)$$

$$\begin{aligned}\Psi_{S,5,c_1} = &(\epsilon_5^{c_1 K} + \epsilon_5^{-c_1 K})\Big\{\beta_1^{K^2} + (-1)^\chi (X_+^{K^2} + X_-^{K^2})\Big\} \\ &+ (\epsilon_5^{2c_1 K} + \epsilon_5^{-2c_1 K})\Big\{\beta_2^{K^2} + (-1)^\chi (Y_+^{K^2} + Y_-^{K^2})\Big\} \\ &+ \Bigg\{\Big(\sqrt{\frac{\alpha}{\beta_1\beta_2}} X_+ Y_+\Big)^{K^2} + \Big(\sqrt{\frac{1}{\alpha\beta_1\beta_2}} X_+ Y_-\Big)^{K^2} \\ &\qquad + \Big(\sqrt{\frac{1}{\alpha\beta_1\beta_2}} X_- Y_+\Big)^{K^2} + \Big(\sqrt{\frac{\alpha}{\beta_1\beta_2}} X_- Y_-\Big)^{K^2}\Bigg\}\end{aligned}$$

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\end{aligned}
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where $X_\pm$, $Y_\pm$ are solutions of

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X^2 - \frac{4}{5}\beta_1(\beta_1 t_1 - 1)(3\rho^{-5} + 1)X + \frac{4}{5}\beta_1^2(3\rho^{-5} + 1) = 0
$$

$$
Y^2 - \frac{4}{5}\beta_2(\beta_2 t_2 - 1)(1 - 3\rho^{5})Y + \frac{4}{5}\beta_2^2(1 - 3\rho^{5}) = 0
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Suppose $\mathbb{E}$ universal sheaf on $M \times S$

$$\tau_\alpha(\sigma) := \pi_{M*}\big(\pi_S^*\sigma \cap \mathrm{ch}_\alpha(\mathbb{E})\big) \in H^*(M, \mathbb{Q}), \quad \forall \alpha \geq 0, \sigma \in H^*(S, \mathbb{Q})$$

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Donaldson invariants: $\int_{[M]^{\mathrm{vir}}} P(\mathbb{E})$, $\quad P(\mathbb{E})$ any poly. in $\tau_\alpha(\sigma)$'s

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$\int_{[M]^{\mathrm{vir}}} P(\mathbb{E}) =$ "formula in terms of $\int_{\mathrm{Hilb}^{n_1}(S)\times\cdots\times\mathrm{Hilb}^{n_r}(S)} \cdots$"

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- Calculate integrals over products of Hilbert schemes
- Check conjectures for $r = 2, 3$ for list of S, H, c_1, c_2

Vafa-Witten partition function 1994 (assume: r prime)

$$\mathsf{Z}_{r,c_1}^{\mathrm{VW}}(q) \approx q^{\bullet} \sum_{c_2} e(M_S^H(r, c_1, c_2))\, q^{c_2} + \underbrace{\mathsf{Z}_{r,c_1}^{\mathrm{vert}}(q)}_{???}$$

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Consider Higgs moduli space $N_S^H(r,c_1,c_2)$

Higgs pairs $(\mathcal{E},\phi)$

- $\mathcal{E}$ torsion free sheaf, $\mathrm{rk}(\mathcal{E}) = r$, $c_i(\mathcal{E}) = c_i$
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i.e. $\forall$ ϕ-invariant $0 \neq \mathcal{F} \subsetneq \mathcal{E}$: $\frac{\chi(\mathcal{F}(mH))}{\mathrm{rk}(\mathcal{F})} < \frac{\chi(\mathcal{E}(mH))}{\mathrm{rk}(\mathcal{E})}$ for all $m \gg 0$

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$$N_S^H(r,c_1,c_2) := \Big\{(\mathcal{E},\phi) \,:\, \text{stable Higgs pair}, \mathrm{rk}(\mathcal{E}) = r, c_i(\mathcal{E}) = c_i\Big\}/\cong$$

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Define $Z_{r,c_1}^{\mathrm{VW}}(q) \overset{\mathrm{TT}}{:=} q^{\bullet} \sum_{c_2} q^{c_2} \int_{[N_S^H(r,c_1,c_2)]^{\mathrm{vir}}} 1$

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S-duality conjecture Vafa-Witten, 1994

$$\mathsf{Z}^{\mathrm{VW}}_{r,c_1}(-1/\tau) = \left(\frac{r\tau}{i}\right)^{-\frac{e(S)}{2}} \sum_{w \in H^2(S,\mathbb{Z}_r)} \epsilon_r^{c_1 w}\, \mathsf{Z}^{\mathrm{VW}}_{r,w}(\tau), \quad q = e^{2\pi i \tau}$$

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$$\overset{\text{roughly}}{\Rightarrow} \mathsf{Z}^{\mathrm{vert}}_{r,c_1}(q) \xleftrightarrow{\tau \mapsto -1/\tau} q^\bullet \sum_{c_2} e^{\mathrm{vir}}(M_S^H(r, c_1, c_2))\, q^{c_2}$$

$r = 2, 3, 5$: conjectures $\mathsf{Z}^{\mathrm{vert}}_{r,c_1}(q) \Rightarrow$ conjectures $e^{\mathrm{vir}}(M_S^H(r, c_1, c_2))$