Waring problems and the Lefschetz properties

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Waring problem in number theory

The Waring problem, in number theory, is the search, for each exponent k, for the minimum s such that every positive integer can be decomposed as a sum of at least s perfect k-th powers.

The first result in this direction was proved by Lagrange. Any positive integer is the sum of four squares. Any positive integer is the sum of nine cubes, 19 fourth powers, 37 fifth powers and 73 sixth powers.

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Waring problem for polynomials

In analogy, the algebraic Waring problem asks what is the minimum s such that any homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]_d$, of degree d, can be decomposed as a sum of at least s perfect d-th powers of linear forms.

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Variants of the Waring problem

We are interested in three variants of the Waring problem, our focus are special forms. In order to change to a local problem, we consider these notions of rank for $f \in R_d = \mathbb{C}[x_0, \dots, x_n]_d$.

1. The *Waring rank* of f is its algebraic rank: it is the minimum $s = wrk(f)$ such that f can be decomposed as a sum of d-th powers of s distinct linear forms.
2. The *Border rank* of f is its geometric rank: it is the minimum $s = \underline{rk}(f)$ such that the class $[f] \in \mathbb{P}(R_d)$ belongs to the s-th secant variety of the Veronese variety $\mathcal{V}_d(\mathbb{P}^n) \subset \mathbb{P}(R_d)$. It is equivalent to say that there is a one parameter family of forms f_t of Waring rank s such that $f = \lim_{t \to 0} f_t$.
3. The *Cactus rank* of f is its schematic rank: it is the minimum $s = cr(f)$ such that there is a finite scheme K of length s, $K \subset \mathcal{V}_d(\mathbb{P}^n) \subset \mathbb{P}(R_d)$ such that $[f] \in < K >$.

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Wild forms

We know that $\underline{rk}(f) \leq wrk(f)$ and $cr(f) \leq wrk(f)$, while in general $cr(f)$ and $\underline{rk}(f)$ are incomparable. Very few examples are known satisfying $cr(f) > \underline{rk}(f)$, they are called *wild forms*. According to the best of our knowledge, the first example of wild form was constructed by W. Buczyńska, J. Buczyński.

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Consider the cubic $f = xu^2 + y(u+v)^2 + zv^2 \in \mathbb{C}[x,y,z,u,v]_3$. It is easy to compute its Waring rank $wrk(f) = 9$. They showed, explicitly, that $\underline{rk}(f) \leq 5$. Indeed, $f = \lim_{t \to 0} \frac{1}{t} f_t$, with

$$36 f_t = 12(u+tx)^3 - 12(u+v+ty)^3$$

$$-4(2v-tz)^3 - 4(u-v)^3 + 4(u+2v)^3.$$

On the other hand, $cr(f) = 6$ which agrees with the description of f as the sum of three double points in the Veronese.

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The hypersurface $X = V(f) \subset \mathbb{P}^4$ given by $f = xu^2 + y(u+v)^2 + zv^2 \in \mathbb{C}[x,y,z,u,v]_3$ appears in the 1901 paper of Perazzo, since it is a counter-example of Hesse's claim that all forms with vanishing hessian are cones.

Recalling Gordan-Noether criterion, the vanishing of the Hessian is equivalent to the non dominance of the gradient map $\nabla_f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$. In fact, the hessian is the Jacobian of the gradient. Since $4f_xf_z = (f_y - f_x - f_z)^2$, the gradient map is not dominant. The surprising fact is that the vanishing of the Hessian is also related with the annihilator of f in the sense of Macaulay-Matlis duality.

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Forms with minimal border rank and vanishing Hessian

This idea was generalized for concise forms with minimal border rank and vanishing Hessian by Huang, Michałek and Ventura.

Theorem (Huang, Michałek and Ventura)

Let $f \in R_d$ be a concise form with minimal border rank. If $\text{hess}_f = 0$, *then f is wild.*

Macaulay-Matlis duality

Theorem (Double annihilator Theorem of Macaulay)

Let $R = \mathbb{K}[x_0, x_1, \ldots, x_n]$ *and let* $Q = \mathbb{K}[X_0, X_1, \ldots, X_n]$ *be the ring of differential operators. Let* $A = \bigoplus_{i=0}^{d} A_i = Q/I$ *be a standard graded Artinian* $\mathbb{K}$*-algebra. Then* A *is a standard graded Gorenstein algebra of socle degree* d *if and only if there exists* $f \in R_d$ *such that* $A \simeq Q/\operatorname{Ann}(f)$*.*

We say that f is concise if $\dim A_1 = n + 1$, or equivalently $I_1 = 0$. In this case *codim*$A = n + 1$.

Mixed Hessian

Let $A = Q/\operatorname{Ann}(f)$ be a standard graded Artinian Gorenstein K-algebra of socle degree d. Let $k \leq l \leq d$ be two integers and let $B_k = (\alpha_1, \ldots, \alpha_{m_k})$ be an ordered $\mathbb{K}$-linear basis of A_k and let $B_l = (\beta_1, \ldots, \beta_{m_l})$ be an ordered $\mathbb{K}$-linear basis of A_l.

The *mixed Hessian* of f of order (k, l) with respect to the basis B_k and B_l is the matrix $\operatorname{Hess}_f^{(k,l)} := [\alpha_i \beta_j(f)]_{m_k \times m_l}$. Moreover, we define $\operatorname{Hess}_f^k = \operatorname{Hess}_f^{(k,k)}$ and $\operatorname{hess}_f^k = \det(\operatorname{Hess}_f^k)$ the Hessian matrix of k-th order and the Hessian of k-th order of f respectively. Notice that $\operatorname{hess}_f = \operatorname{hess}_f^1$.

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Hessian criteria for SLP and WLP

Theorem (-,Zappala,2018)

Let $A = Q/\operatorname{Ann}_Q(f)$ be a n AG algebra and $L \in A_1$. Let B_k and B_l be ordered basis ofr A_k and A_l. The matrix of the map $\bullet L^{l-k} : A_k \to A_l$, for $k < l \leq \frac{d}{2}$, with respect to the basis B_k and B_l coincides with $\operatorname{Hess}_f^{(d-l,k)}(L^{\perp})$, using basis B_l^* and B_k. In particular:

$$\operatorname{rk}\left(\bullet L^{l-k}\right) = \operatorname{rk}\left(\operatorname{Hess}_f^{(d-l,k)}(L^{\perp})\right).$$

An upper bound for the border rank of GNP

We recall that a form is wild if $cr(f) > \underline{rk}(f)$. Our strategy to construct wild forms is to find an upper bound for the border rank and a lower bound for the cactus rank and compare them.

Proposition

Let $f \in \mathbb{C}[x_1, \ldots, x_n, u, v]_{(k,d-k)}$ be a bi-homogeneous form of bi-degree $(k, d-k)$ with $1 \leq k \leq d-k$. The border rank of f satisfies:

$$\underline{rk}(f) \leq k(d+2).$$

Degenerated Hessian and saturation

A form $f \in R_d$ is called *k-concise*, with $d \geq 2k+1$, if $I_j = 0$ for $j = 1, 2, \ldots, k$. It is equivalent to $a_j = \binom{n+j}{j}$ for $j = 0, \ldots, k$. As usual, 1-concise forms are called concise.

Lema

Let $f \in R_d$ be a k-concise form and $A = Q/I$ be the associated algebra with $I = \mathrm{Ann}_f$. Suppose that $a_k \leq a_{d-s}$ and $k + s \leq d$. If $\mathrm{Hess}_f^{(k,s)}$ is degenerated, then exists $\alpha \in I_k^{sat} \setminus I_k$.

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Proof

We are considering $\text{Hess}_f^{(k,s)}$ as a matrix in R. By the Hessian criteria 3, for each $L \in A_1$, the map $\bullet L^{d-s-k} : A_k \to A_{d-s}$ is represented by $\text{Hess}_f^{(k,s)}(L^\perp)$. Therefore, there is a universal polynomial in the kernel of $\text{Hess}_f^{(k,s)}$ such that its image $\alpha \in A_k$ belongs the kernel of $\bullet L^{d-s-k}$ for every $L \in A_1$, that is $L^{d-s-k}\alpha \in I_{d-s}$. In particular, $X_i^{d-k-s}\alpha \in I_{d-s}$ for $i = 0, \dots, n$, that is, $\alpha \in I_k^{sat} \setminus I_k$.

k-concise wild forms with vanishing Hessian

Lema

Let $f \in R_d$ be a $k-$concise form with $2k < d$ and let $I = \mathrm{Ann}(f) \subset Q$. Let $J = (I_{d-k}) \subset Q$ be the ideal generated by the degree $d-k$ part of I. If $J_l^{sat} \neq \emptyset$ for some $l \leq k$, then

$$cr(f) > a_k = \binom{n+k}{k}.$$

k-concise wild forms with vanishing Hessian

Theorem

Let $f \in R_d$ be a k-concise homogeneous form, with $2k \leq d$. If $\text{hess}_f = 0$, *then*

$$cr(f) > \binom{n+k}{k}.$$

In particular, if $\underline{rk}(f) \leq \binom{n+k}{k}$, then f is wild.

k-concise wild forms with vanishing Hessian

The following Corollary is the main resul of Huang, Michałek and Ventura.

Corollary

Let $f \in R_d$ be a concise form with minimal border rank. If $\text{hess}_f = 0$, *then f is wild.*

Proof.

Minimal border rank means $\underline{rk}(f) = n + 1$. Since f is 1-concise and $\text{hess}_f = 0$, by Theorem 7, we get $cr(f) > n + 1$. □

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Example (A wild form with non minimal border rank)

Consider the forms $f \in \mathbb{C}[x, y, z, u, v]_{288}$, given by $f = g^{16}$ with

$$f = xu^{17} + yu^{16}v + zv^{17}.$$

We know that f has vanishing Hessian. Indeed, by Gordan-Noether criteria, since the partial derivatives of g satisfy $g_x^{16} g_z = g_y^{17}$, they are algebraically dependent, therefore, $\operatorname{hess} f = 0$. Moreover, the choice of g was in such a way that its polar image has degree d. If the polar degree was lower, then the f could be not 16-concise. We checked the 16-conciseness of f which implies that its border rank is non minimal. In this case $cr(f) > a_{16} = 4845$ and $\underline{rk}(f) \leq 4640$, hence f is wild.

k-concise wild forms with non vanishing Hessian

Lema

Let $f \in R_d$ be a $k-$concise form with $2k < d$. Let $I = \mathrm{Ann}(f) \subset Q$ and $A = Q/I$. Suppose that Hilb(A) *is unimodal. Let $J = (I_{\leq d-k}) \subset Q$ be the ideal generated by the graded parts of degree $\leq d-k$ of I. If $J_l^{sat} \neq \emptyset$ for some $l \leq k$, then*

$$cr(f) > a_k = \binom{n+k}{k}.$$

k-concise wild forms with non vanishing Hessian

Theorem

Let $f \in R_d$ be a k-concise homogeneous form with $2k \leq d$ and let l, s be integers such that $l \leq k \leq s$ and $s + l \leq d$. Let $I = \text{Ann}(f)$ and $A = Q/I$ and suppose that $\text{Hilb}(A)$ is unimodal. Suppose that $\text{Hess}_f^{(l,s)}$ is degenerated, or equivalently, for a generic $L \in A_1$, the map $\bullet L^{d-s-l} : A_l \to A_{d-s}$ is not injective. Then:

$$cr(f) > \binom{n+k}{k}.$$

In particular, if $\underline{rk}(f) \leq a_k$, then f is wild.

Sketch of the Proof

Let $I = \mathrm{Ann}_f$ and consider the algebra $A = Q/I$. Let $a_i = \dim A_i$. Since A is Gorenstein and k-concise, we get $a_k = a_{d-k} = \binom{n+k}{k}$, by Poincaré duality. Let $J = (I_{\leq d-k})$ be the ideal generated by the pieces of I in degree $\leq d-k$. Let $B = Q/J$ and $b_i = \dim B_i$, we get that $b_k = \binom{n+k}{k}$ and $b_{d-k} = a_{d-k}$. By hypothesis we have

$$a_l = b_l \leq a_k = b_k \leq a_s = b_s = a_{d-s} = b_{d-s}.$$

By Lemma 5, there is $\gamma \in I_l^{sat}$. By hypothesis $s \geq k$, therefore, $d-s \leq d-k$, which implies $I_{d-s} = J_{d-s}$, hence $\gamma \in J_l^{sat}$. The result follows from Lemma 10.

The first example of a form with vanishing second Hessian whose Hessian is non vanishing was given by Ikeda.

Example (A wild form without vanishing hessian)

Let $f = xu^3v + yuv^3 + x^2y^3 \in \mathbb{C}[x, y, u, v]_5$. Let $A = Q/\operatorname{Ann}_f$, we get

$$\operatorname{Hilb}(A) = (1, 4, 10, 10, 4, 1).$$

Therefore f is 2-concise. We know that $\operatorname{hess}_f^2 = 0$. By Proposition 4, $\underline{rk}(xu^3v + yuv^3) \leq 7$. We know that, $\underline{rk}(x^2y^3) = 3$, then $\underline{rk}(f) \leq 10$. By Theorem 11 we get that $cr(f) > 10$, therefore f is wild.

Corollary

Let $M_i \in \mathbb{C}[x_0, \dots, x_n]_k$ *with* $i = 0, \dots, b-1$ *be all the monomials of degree* k, *where* $b = \binom{n+k}{k}$. *Let*

$$f = \sum_{i=0}^{b-1} M_i u^{b-i} v^i \in \mathbb{C}[x, y, z, u, v]_{b+k}.$$

If $\binom{n+k+2}{k} > k[(k+1) + \binom{n+k}{k}]$, *then* f *is wild.*