Quiver representations & multiple flag varieties

↳ a panoramic survey of joint work with
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presented by Ryan Kinser (U. Iowa)

Talk plan

I. Quiver representation varieties & multiple flag varieties — 30 min

~~~ 5 min questions or break ~~~

II. Connections between the two — 30 min

~~~ 5 min questions or break ~~~

III. Applications: combinatorics, singularities, & equivariant K-theory.

Notation and conventions

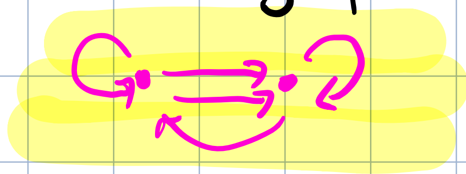
$\mathbb{k}$ is an infinite field

vector spaces are finite dimensional over $\mathbb{k}$.

I.A. Quivers & representation varieties

A quiver Q is a finite directed graph.

- Q_0 : vertex set of Q
- Q_1 : arrow set of Q



A representation V of Q is an assignment of:

- a vector space V_x to each $x \in Q_0$
- a linear map $V_i \xrightarrow{V_a} V_j$ to each $i \xrightarrow{a} j \in Q_1$
- $\mathrm{rep}(Q)$ is the category of reps of Q.

Historical remark: Quiver representations were introduced to study representations of algebras.

P. Gabriel, 1970s

Suppose $\Bbbk = \overline{\Bbbk}$ and let A be a finite-dimensional, associative $\Bbbk$-algebra. Then there exists a quiver Q such that

full subcategory

$$\text{mod-}A \simeq \mathrm{rep}(Q,R) \subseteq \mathrm{rep}(Q)$$

(R is a set of "relations" on Q)

Given V, choose basis to identify:

- $V_x \simeq \Bbbk^{\alpha_x}$ for some $\alpha = (\alpha_x \in \mathbb{Z}_{\geq 0})_{x \in Q_0}$, the dimension vector of V
- $\Bbbk^{\alpha_i} \simeq V_i \xrightarrow{V_a} V_j \simeq \Bbbk^{\alpha_j}$ given by an $\alpha_i \times \alpha_j$ matrix (acting on row vectors)

Example:

$$Q = \begin{tikzcd} \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet \arrow[d] & \\ & \bullet & \end{tikzcd} \rightsquigarrow V = \begin{tikzcd} \Bbbk \arrow[dr, "{[1\,0]}"'] & & \Bbbk \arrow[dl, "{[0\,1]}"] \\ & \Bbbk^2 \arrow[d, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] & \\ & \Bbbk & \end{tikzcd} \in \mathrm{rep}(Q)$$

$$\alpha = \begin{smallmatrix} 1 & & 1 \\ & 2 & \\ & 1 & \end{smallmatrix}$$

If we instead fix α, and let the matrices vary arbitrarily of fixed size, we get the representation variety

$$\mathrm{rep}(Q,\alpha) := \prod_{i \xrightarrow{a} j \in Q_1} \underbrace{\mathrm{Mat}_{\alpha_i \times \alpha_j}(\Bbbk)}_{V_i^* \otimes V_j}$$

variety or vector space

Example: Keep Q as above with $\alpha = \begin{smallmatrix} 1 & & 2 \\ & 7 & \\ & 5 & \end{smallmatrix}$

$$\mathrm{rep}(Q,\alpha) = \mathrm{Mat}_{1\times 7} \times \mathrm{Mat}_{2\times 7} \times \mathrm{Mat}_{7\times 5} \simeq \mathbb{A}^{56}$$

$$\boxed{\text{The equivariant geometry is what is interesting!}}$$

The base change group acting on $\mathrm{rep}(Q,\alpha)$ is:

$$GL(\alpha) := \prod_{x \in Q_0} GL(\alpha_x)$$

Example: Continuing with $\alpha = \begin{tikzcd}[row sep=small, column sep=small] 1 \arrow[dr] & & 2 \arrow[dl] \\ & 7 \arrow[d] & \\ & 5 & \end{tikzcd}$, we have

$GL(\alpha) = GL(1) \times GL(2) \times GL(7) \times GL(5)$, and

$g = \begin{matrix} g_1 & & g_2 \\ & g_3 & \\ & g_4 & \end{matrix} \in GL(\alpha)$ acts on $V = \begin{tikzcd}[column sep=small] \mathbb{k} \arrow[dr, "V_a"'] & & \mathbb{k}^2 \arrow[dl, "V_b"] \\ & \mathbb{k}^7 \arrow[d, "V_c"] & \\ & \mathbb{k}^5 & \end{tikzcd} \in rep(Q,\alpha)$

by $V \cdot g = \begin{tikzcd}[column sep=small] \mathbb{k} \arrow[dr, "g_1^{-1} V_a g_3"'] & & \mathbb{k}^2 \arrow[dl, "g_2^{-1} V_b g_3"] \\ & \mathbb{k}^7 \arrow[d, "g_3^{-1} V_c g_4"] & \\ & \mathbb{k}^5 & \end{tikzcd}$

Easy but crucial fact: for $V, W \in rep(Q)$

$V \simeq W$ in $rep(Q)$ $\Longleftrightarrow$ V, W are in the same $GL(\alpha)$-orbit of $rep(Q,\alpha)$.

rep theory   geometry

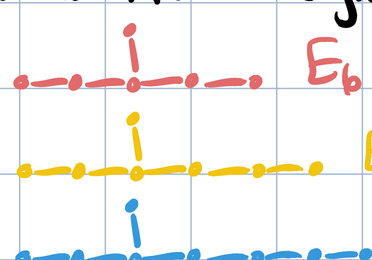
(1972)

Gabriel's Theorem Let Q be a connected quiver.

$rep(Q,\alpha)$ has finitely many $GL(\alpha)$-orbits

$\Updownarrow$

the underlying graph of Q is ADE Dynkin

 or  or 

type A   type D E_6, E_7, E_8

I. B. Multiple flag varieties

A (complete) flag in $\mathbb{k}^n$ is a chain of subspaces:
$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{k}^n, \qquad \dim V_i = i$$

A flag can be represented by $M \in GL(n)$ where V_i is the span of the first i rows of M.

Example $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ represents the standard flag $0 < \langle e_1 \rangle < \langle e_1, e_2 \rangle < \mathbb{k}^3$.

Also $\begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & 0 \\ 4 & 7 & 3 \end{bmatrix}$ represents the same flag.

Let
$$B_- = \begin{bmatrix} * & & & & & 0 \\ * & * & & & & \\ * & * & * & & & \\ * & * & * & * & & \\ * & * & * & * & * & \\ * & * & * & * & * & * \end{bmatrix} < GL(n)$$
be the subgroup of lower-triangular matrices.

Generalizing the example, flags in $\mathbb{k}^n$ are in bijection with cosets $B_- g \in B_- \backslash GL(n)$

The quotient space $\mathrm{Flag}(n) := B_- \backslash GL(n)$ is a projective variety called a flag variety. Its points are in bijection with flags in $\mathbb{k}^n$.

- We can also consider partial flags $0 = V_0 \subset V_1 \cdots \subset V_r = \mathbb{k}^n$, $\dim V_i = d_i$ & get $\mathrm{Flag}(d_1, \ldots, d_r) =: P_- \backslash GL(n)$ where P_- is block lower triangular. P_- depends on $(d_1, \ldots, d_r)$.

2 5 7

- The standard group we consider acting on flag varieties is Upper-triangular matrices B, by right mult. on matrices.

- Multiple flag varieties are products of flag varieties, with B acting diagonally.

↳ Key example: double Grassmannians

$$\mathrm{Flag}(a,n) \times \mathrm{Flag}(b,n) =: Gr(a,n) \times Gr(b,n)$$

$$b \cdot (x, y) = (bx, by)$$

- Our work focuses on the following problems for quiver rep varieties:
  - types of singularities in orbit closures
  - partial order on orbit closures (combinatorial)
  - polynomial representatives of equivariant Grothendieck classes of orbit closures.

They are better understood for multiple flag varieties... so we want connections.

II. Connections

In this section, we give explicit constructions which realize our problems for certain classes of varieties above as "specializations" of the same problems for other classes.

II.A. The "easy" connections

① Partial flag varieties as "specializations" of type A quiver rep varieties

Given $X = \mathrm{Flag}(d_1, d_2, \dots d_r = n)$

consider the type A quiver & dim vector:

$$(Q,\alpha): d_1 \to d_2 \to d_3 \to \cdots \to n \leftarrow n-1 \leftarrow \cdots 3 \leftarrow 2 \leftarrow 1$$

Let $U = \mathrm{rep}^\circ(Q,\alpha)$ be the open subvariety where all maps are injective.

$G = GL(n)$, $G' = GL(d_1) \times \cdots \times GL(d_{r-1}) \times GL(n-1) \times \cdots \times GL(1)$.

Note $GL(\alpha) = G \times G'$.

At the level of sets, easy to see G' acts freely on U, and

$$\{G' \text{ orbits on } U\} \longleftrightarrow \{(\text{partial flag, complete flag}) \text{ pairs}\}$$

In fact we have a geometric quotient

$$U/G' \simeq X \times (B\backslash G) \simeq X *_B G$$

which is G-equivariant.

homogeneous fiber bundle h.f.b.

Using general properties of h.f.b.'s & GIT, problems about B-orbits on X are specializations of problems about $GL(\alpha)$-orbits on $U \underset{\text{open}}{\subset} \operatorname{rep}(Q,\alpha)$.

② Similar story for double Grassmannians.

Given $Y := \underset{V_1 \subset k^n}{Gr(a,n)} \times \underset{V_2 \subseteq k^n}{Gr(b,n)}$, take

$$(Q,\alpha): \begin{array}{l} a \searrow \\ b \nearrow \end{array} n \leftarrow n-1 \leftarrow \cdots \leftarrow 2 \leftarrow 1$$

type D

and $G = GL(n)$ & $GL(\alpha) = G \times G'$.
With the same definitions above, we again get

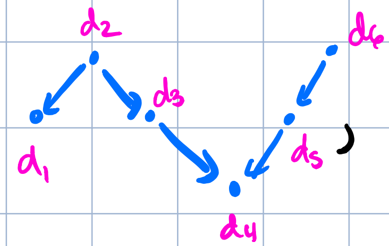
$$G\text{-equivariant}: \ U/G' \simeq Y *_B G$$

The core ideas of this "easy direction" have been around for decades.
Details in: [KR20, Thm. 2.6]

II. B. Type A quiver reps inside mutiple flag varieties

Based on [KR 15], inspired by works of Zelevinsky (1985) and Lakshmibai - Magyar (1998)

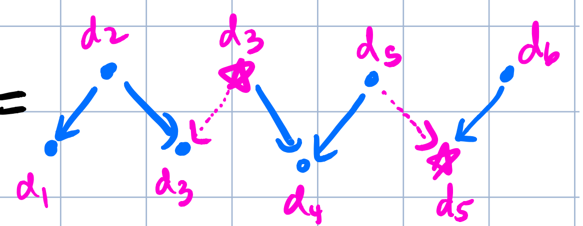
$\bullet \to \bullet \to \bullet \to \cdots \to \bullet$

Given a type A quiver Q, say 

,
and dimension vector α,
we want to find $\mathrm{rep}(Q,\alpha)$ "inside" a partial flag variety.

But where is the flag?

<u>Step 1</u>: inflate (Q,α) to bipartite setting

$(\tilde{Q},\tilde{\alpha}) =$ 

Let $\mathrm{rep}^\circ(\tilde{Q},\tilde{\alpha})$ be open subvariety where matrices over newly added arrows are invertible.

<u>Prop</u>: There is a $GL(\tilde{\alpha})$-equivariant isomorphism:

$$\mathrm{rep}^\circ(\tilde{Q},\tilde{\alpha}) \simeq \mathrm{rep}(Q,\alpha) *_{GL(\alpha)} GL(\tilde{\alpha}).$$

h.f.b.

So by properties of h.f.b.s, equivariant geometry for type A quivers in arbitrary orientation can be reduced to bipartite orientation.

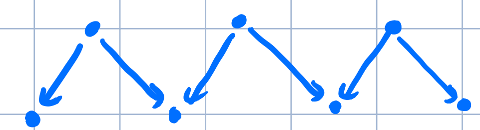
Step 2: the bipartite Zelevinsky map

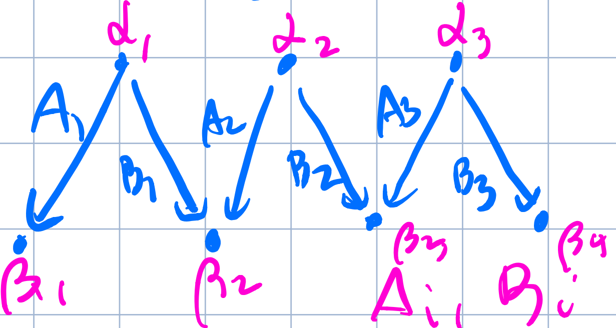
Let Q be a bipartite type A quiver with dim vec α. In [KR15] we constructed a morphism of varieties

$$\zeta : \operatorname{rep}(Q,\alpha) \hookrightarrow \operatorname{Flag}(d_1,\dots,d_r) =: X$$

such that for each orbit closure $\Omega \subseteq \operatorname{rep}(Q,\alpha)$, $\zeta(\Omega)\times \mathbb{A}^N$ is an open subvariety of a $\underbrace{B\text{-orbit closure in } X}_{\text{Schubert variety}}$.

Example Consider the bipartite type A quiver rep:

$Q =$ 

$V =$ 

A_i, B_i matrices

Then $\zeta(V)$ is the flag represented by the matrix:

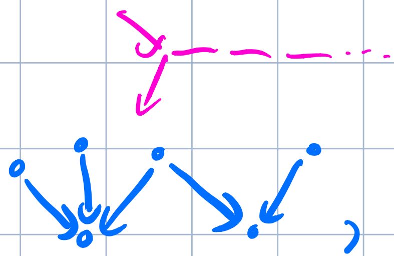
$\alpha_2 = \beta_1+\beta_2$, $\alpha_1 = \beta_1$

$$\zeta(V) = \begin{array}{c} \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \end{array} \begin{array}{c} \begin{matrix} \alpha_3 & \alpha_2 & \alpha_1 & & & & \end{matrix} \\ \left[\begin{array}{ccc|cccc} 0 & 0 & A_1 & 1 & & & \\ 0 & A_2 & B_1 & & 1 & & \\ A_3 & B_2 & 0 & & & 1 & \\ B_3 & 0 & 0 & & & & 1 \\ \hline 1 & & & & & & \\ & 1 & & & & 0 & \\ & & 1 & & & & \end{array}\right] \end{array} \in \operatorname{Flag}(d_1,\dots,d_r)$$

1 identity matrices

II.C Type D quiver reps inside double Grassmannians

Based on [KR 20], inspired by work of Bobiński–Zwara (2002).

Given a type D quiver, say ,
we want to find each $\mathrm{rep}(Q,\alpha)$ "inside" a double Grassmannian.

↳ Reduce to bipartite orientation again.

In [KR15] we constructed a morphism of varieties

$$\zeta : \mathrm{rep}(Q,\alpha) \hookrightarrow Gr(a,n) \times Gr(b,n) =: Y$$

such that for each orbit closure $\Omega \subseteq \mathrm{rep}(Q,\alpha)$,
$\zeta(\Omega) \times Z$ is an open subvariety of a
B-orbit closure in Y. ⇐ generalized Schubert variety
(where Z is some smooth variety).

Remark: In fact, the image of ζ lands in the subset of Y represented by invertible matrices. It can be identified with the "symmetric variety" G/K where $G = GL(a+b)$ & $K = GL(a) \times GL(b)$.

III. Applications

Viewing various equivariant geometry problems as specializations of others above leads to the following.

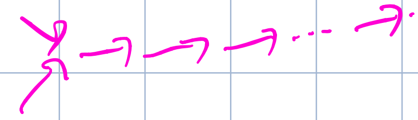
III.A. Orbit closures

"Zelevinsky permutation" of quiver orbit closure

Theorem [KR15]

Let (Q, d) be type A. Then the poset of $GL(d)$-orbit closures in $rep(Q,d)$ embeds as a subposet of the B-orbit closures in a partial flag variety, i.e. a subposet of a symmetric group with Bruhat order.

Remark: A converse statement also holds by the "easy" connections II.A.



Theorem [KR20]

Let (Q, d) be type D. Then the poset of $GL(d)$-orbit closures in $rep(Q,d)$ embeds as a subposet of the B-orbit closures in a symmetric variety $G/K := GL(a+b)/GL(a)\times GL(b)$, i.e. a subposet of clans. [MO90], [S07], [W16].

More detail in joint work in progress with Z. Hamaker & J. Rajchgot

Remark: Similar connecting statements are obtained by the "easy" embedding II.A. & open embedding $G/K \subset Gr(a,n)\times Gr(b,n)$.

III. B. Singularities

Theorem [KR15], [BZ01]
The types of singularities appearing in $GL(d)$-orbit closures in $\mathrm{rep}(Q, \alpha)$, with Q varying over all type A quivers & α over all dim vectors are exactly the singularities appearing in $\underbrace{\text{B-orbit closures in flag varieties}}_{\text{schubert varieties}}$.

They are always normal, Cohen-Macaulay & rational singularities (in char 0)

Theorem [KR20], [BZ02] Based on work of M. Brion
The types of singularities appearing in $GL(d)$-orbit closures in $\mathrm{rep}(Q, \alpha)$, with Q varying over all type D quivers & α over all dim vectors are exactly the singularities appearing in B-orbit closures in double Grassmannians and those in symmetric varieties $GL(a+b)/GL(a) \times GL(b)$.

Again normal, CM, rational resolutions ...

Lőrincz: in type D we find singularities not appearing in type A

III.C. Equivariant K-theory

For an algebraic group G & algebraic G-scheme X, denote by $K_G(X)$ the Grothendieck group of the category of G-equivariant coherent sheaves on X.
The connections of part II. induce maps on these.

Theorem [KR20]
Let $(Q,\mathbf{d})$ be of type D & G/K as above.
Let $T \subset B \subset GL(n)$ & $T(\mathbf{d}) \subset GL(\mathbf{d})$ be maximal tori.
Then the map ζ above induces a homomorphism
$\zeta^* : K_T(G/K) \longrightarrow K_{T(\mathbf{d})}(\mathrm{rep}(Q,\mathbf{d}))$. Laurent polynomial ring

Thus the class $[\Omega] \in K_{T(\mathbf{d})}(\mathrm{rep}(Q,\mathbf{d}))$ can be "computed" by applying ζ^* to the corresponding orbit closure in G/K.
This is useful because latter is better understood, e.g. by [WY14].

Other results in type A on combinatorics & equiv. K-theory joint with A. Knutson & J. Rajchgot.

work in progress on type D with Hamaker & Rajchgot

work in progress with M. Lanini
& J. Rajchgot on
"symmetric quivers" (Derksen–Weyman)
& symmetric quotients $GL(n)/SO(n)$
$GL(2n)/Sp(2n)$

GV Dδ VLLAGT