The Cremona group

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GA20
AG20

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THE CREMONA GROUP

$$\mathrm{Bir}(\mathbb{P}^n) \ := \ \left\{ \varphi : \mathbb{P}^n \ \overset{\sim}{- \, - \!\! \rightarrow} \ \mathbb{P}^n \ \text{ birational self-map } \right\}$$

SETUP

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- Projective spaces:

$$\begin{aligned}\mathbb{P}^n &= \mathbb{C}^{n+1} \setminus \{\bar{0}\} \Big/ \ (x_0, \ldots, x_n) \sim \lambda(x_0, \ldots, x_n),\ \lambda \neq 0 \\ &\ni \ (x_0 : \cdots : x_n)\end{aligned}$$

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- Algebraic sets:

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- Zariski topology: closed sets are the algebraic sets

- Projective varieties: irreducible algebraic sets $X \subset \mathbb{P}^N$

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$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & \mathbb{P}^m \\ (x_0 : \cdots : x_n) & \longmapsto & \big(F_0(x_0, \ldots, x_n) : \cdots : F_m(x_0, \ldots, x_n)\big) \end{array}$$

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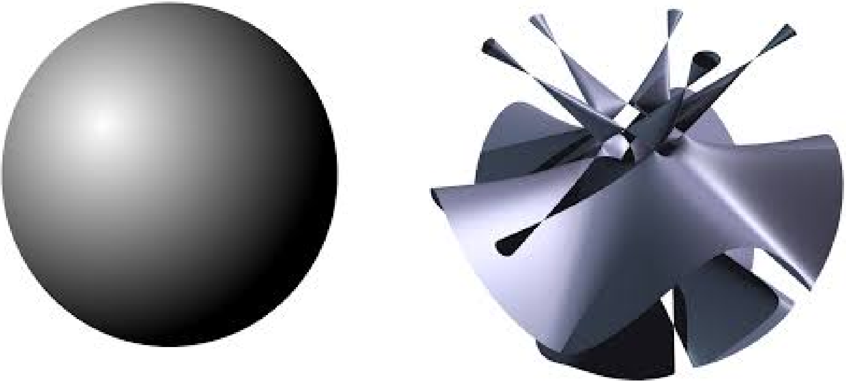
$$\mathbb{C}(\mathbb{P}^n) \ \cong \ \mathbb{C}(x_1, \ldots, x_n)$$

AUTOMORPHISMS OF PROJECTIVE VARIETIES



$X \subset \mathbb{P}^n$ complex projective variety

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EXAMPLE $(X = \mathbb{P}^n)$

$$\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C})$$

$X \subset \mathbb{P}^n$ complex projective variety

$\mathrm{Aut}(X)$ Lie group

$\mathrm{Aut}^0(X) \subset \mathrm{Aut}(X)$ connected component of $\mathbb{I}_X$ ($\mathbb{C}$-algebraic group)

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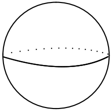
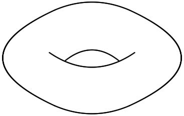
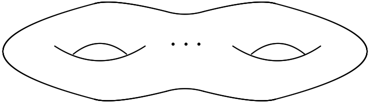
Example (X smooth projective curve of genus g)



$g = 0$



$g = 1$



$g \geq 2$

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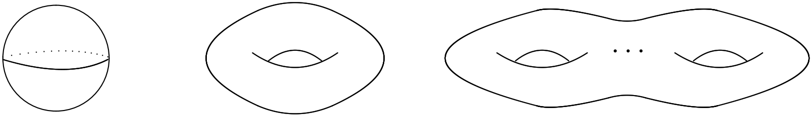

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- $g = 0$ $\mathrm{Aut}^0(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$

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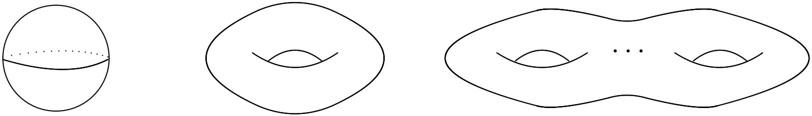

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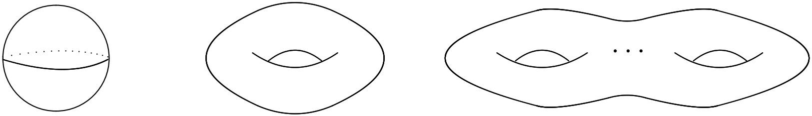
EXAMPLE (X SMOOTH PROJECTIVE CURVE OF GENUS g)



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- $g = 1$ $\mathrm{Aut}^0(X) \cong X$
- $g \geq 2$ $\mathrm{Aut}(X)$ is finite

$X \subset \mathbb{P}^n$ complex projective variety

$$1 \to \underbrace{\mathrm{Aut}^0(X)}_{\mathbb{C}\text{-algebraic group}} \to \mathrm{Aut}(X) \to \underbrace{\mathrm{Aut}(X)/\mathrm{Aut}^0(X)}_{\text{countable discrete group}} \to 1$$

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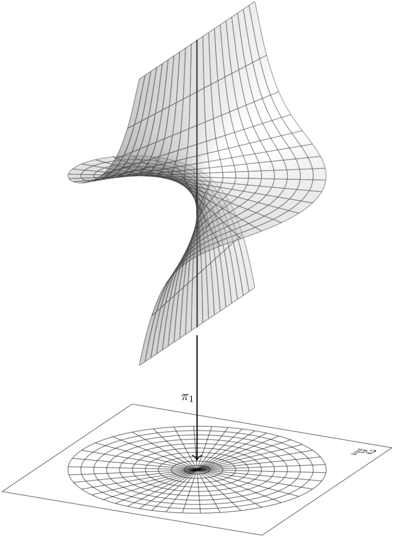
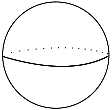
$$1 \to \underbrace{Lin\big(\mathrm{Aut}^0(X)\big)}_{\text{linear algebraic group}} \to \mathrm{Aut}^0(X) \to \underbrace{Ab}_{\text{projective}} \to 1$$

BIRATIONAL GEOMETRY



The classification problem in Algebraic Geometry

In dimension 1 :



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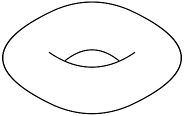

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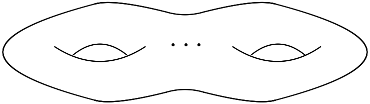

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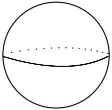

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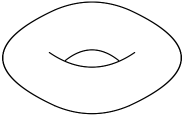

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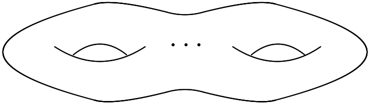

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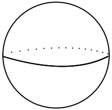

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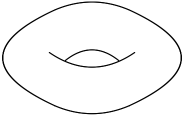

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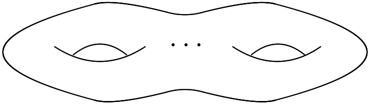

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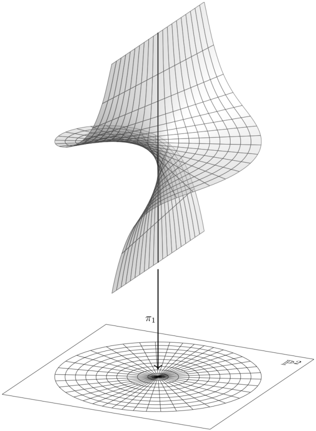
In higher dimensions : too many isomorphism classes

Example (The Blowup of $\mathbb{P}^2$)



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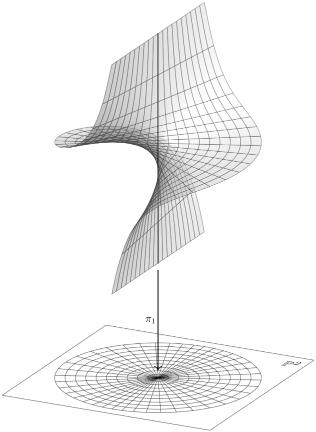

$$P = (0:0:1) \in \mathbb{P}^2$$

$$\begin{tikzcd}
 & \mathbb{P}^2 \times \mathbb{P}^1 & \\
 & \cup & \\
 & \tilde{X} = \overline{\Gamma} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "\pi_P"] & & \mathbb{P}^1 \\
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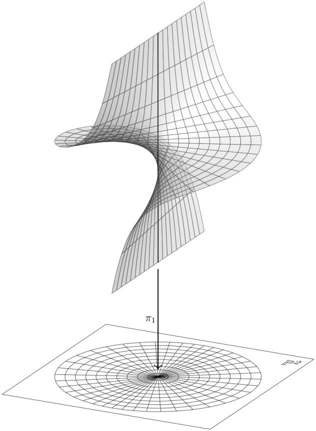

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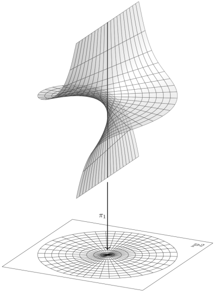
$$\boxed{\pi_1 : \tilde{X} \setminus E \xrightarrow{\cong} \mathbb{P}^2 \setminus \{P\}}$$

$$E = \pi_1^{-1}(P) \cong \mathbb{P}^1$$

The Blowup of $\mathbb{P}^2$ at a point P



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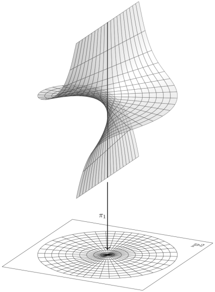

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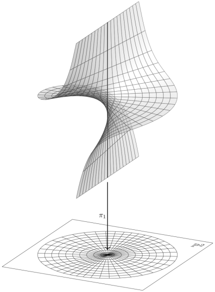

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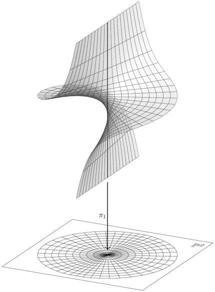

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$$\boxed{\tilde{X} \setminus E \quad \xrightarrow{\cong} \quad X \setminus Z}$$

DEFINITION

X and Y are birational equivalent if $\exists$ dense open subsets $U \subset X$ and $V \subset Y$ and isomorphism

$$X \supset U \xrightarrow{\cong} V \subset Y$$

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- Given a projective variety X, to find a simplest representative in its birational class - minimal model of X

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- Given a projective variety X, to find a simplest representative in its birational class - minimal model of X
- Construct moduli spaces of minimal models (with fixed discrete invariants)

The Minimal Model Program (MMP)

Given a projective variety X, to find a simplest representative in its birational class - minimal model of X

- Dimension 1: Riemann (19^{th} century)
- Dimension 2: Italian school (early 20^{th} century)
- Dimension 3: Mori (1988)
- Dimension ≥ 4: Birkar-Cascini-Hacon-McKernan (2010)

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Problem

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Proof.

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- $p_g(X_d) = 0 \iff \quad d \leq n+1$

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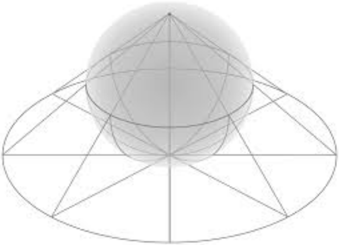
EXAMPLE (QUADRIC HYPERSURFACE $X_2 \subset \mathbb{P}^{n+1}$)



Stereographic projection defines a birational map

$$\pi_P : X_2 \dashrightarrow^{\simeq} \mathbb{P}^n$$

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Is the generic cubic hypersurface $X_3 \subset \mathbb{P}^{n+1}$ rational $(n > 3)$?

Problem

Which properties are invariant under birational equivalence?

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Aut(X) is not a birational invariant

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[d, dashed, "\wr"] & X \arrow[d, dashed, "\wr"] \\
Y \arrow[r, dashed, "\sim"] & Y
\end{tikzcd}$$

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Example (Iskovskikh-Manin 1971)

Any smooth quartic 3-fold $X_4 \subset \mathbb{P}^4$ is irrational

THE CREMONA GROUP

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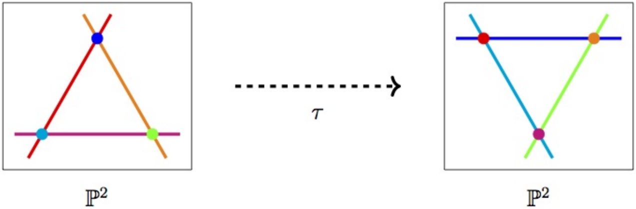

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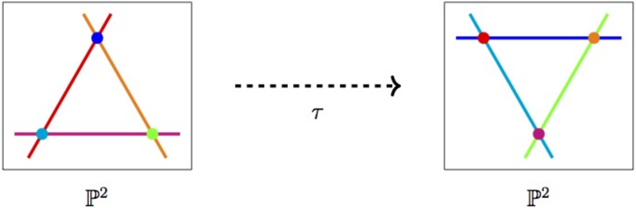

THEOREM (NOETHER-CASTELNUOVO 1870-1901)

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THEOREM (BERTINI 1877, $\cdots$, DOLGACHEV-ISKOVSKIKH 2009)

Classification of finite subgroups of $\mathrm{Bir}(\mathbb{P}^2)$.

The Cremona Group in higher dimensions

THE CREMONA GROUP IN HIGHER DIMENSIONS

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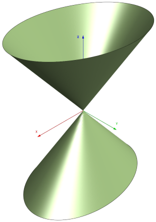

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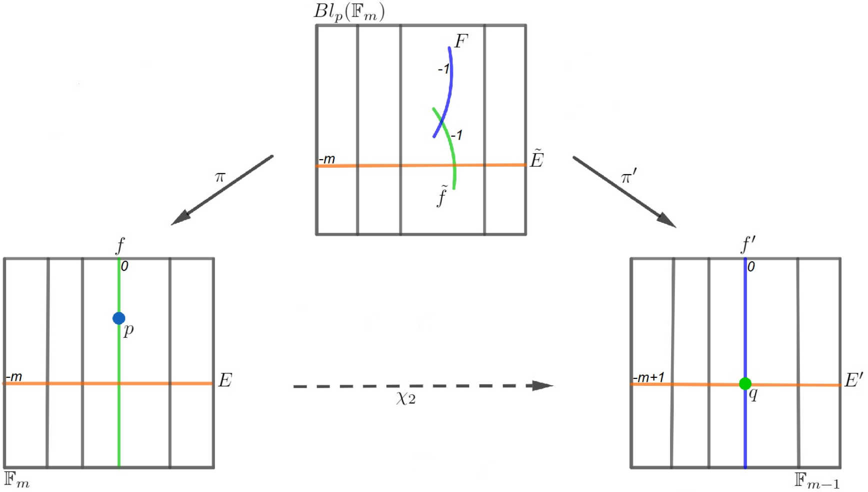
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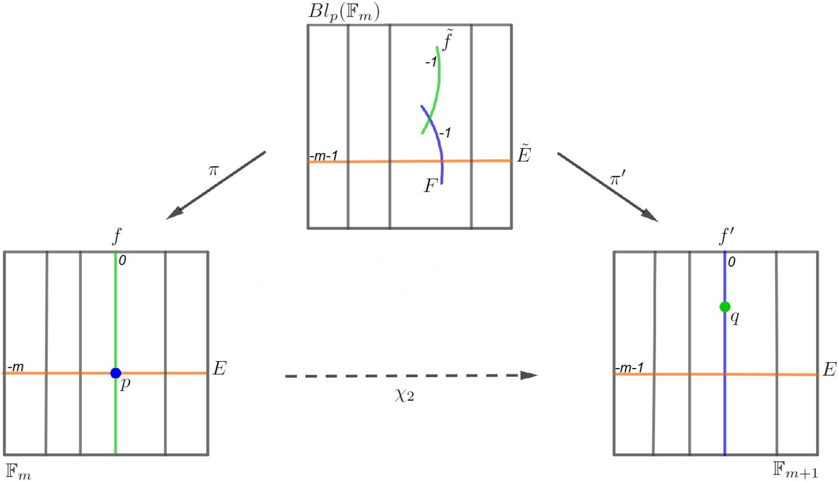
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In both cases, the image of $\mathrm{Aut}(\mathbb{P}^{n+1}, D) \to \mathrm{Aut}(D)$ is finite

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$$\mathrm{Bir}(\mathbb{P}^3, S) \rightarrow \mathrm{Aut}(S)$$

Thank you! [1]

GA20
AG20
DECEMBER 16—18, 2020

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[1] Thanks to Santiago Arango, Daniela Paiva, Charles Staats and Wikipedia for the nice pictures