Gabarito Lista 3

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Exercício 1

a)

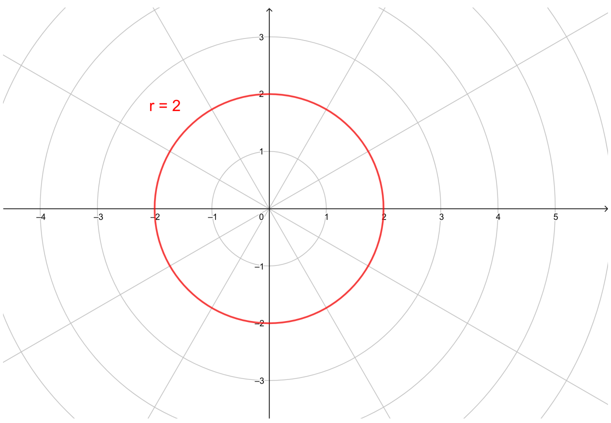

Figure 1: Representação geométrica da curva.

b)

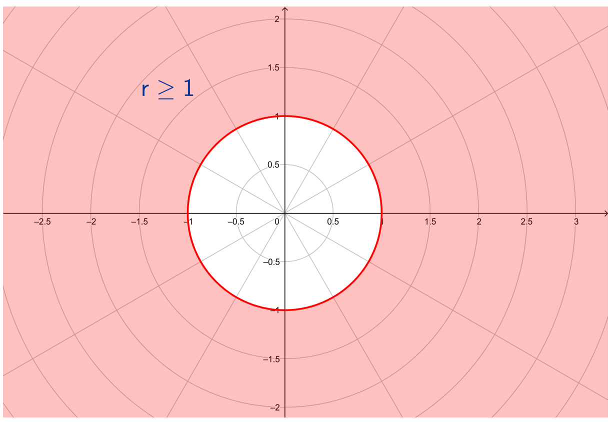

Figure 2: Representação geométrica da curva.

c)

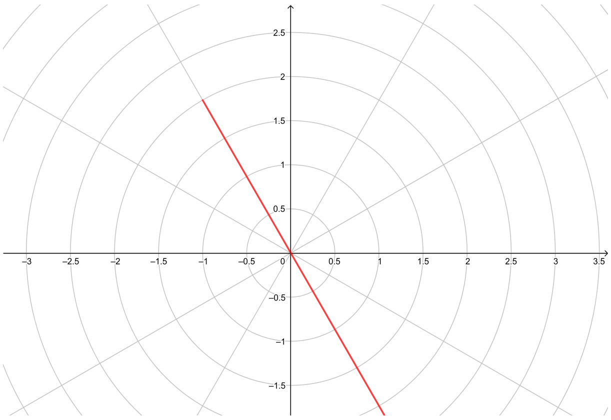

Figure 3: Representação geométrica da curva.

d)

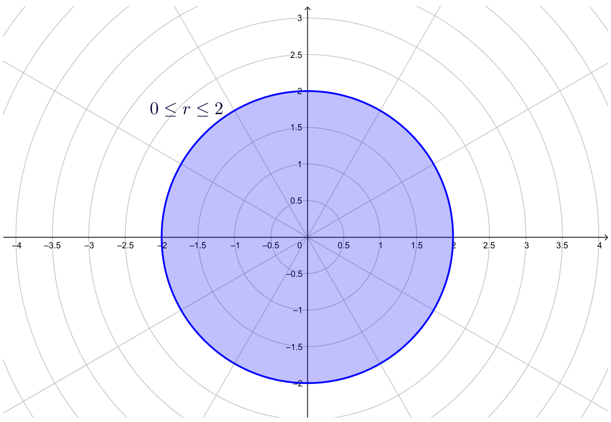

Figure 4: Representação geométrica da curva.

e)

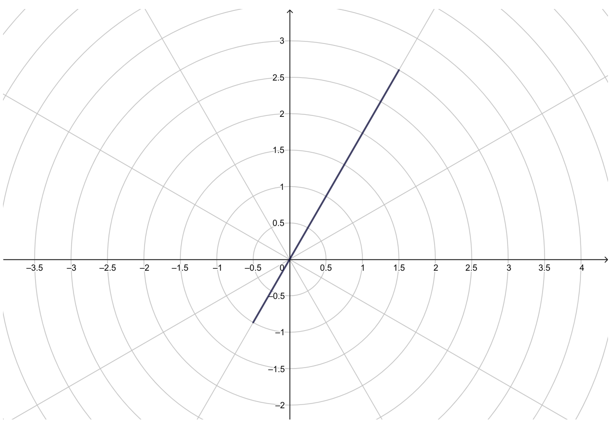

Figure 5: Representação geométrica da curva.

f)

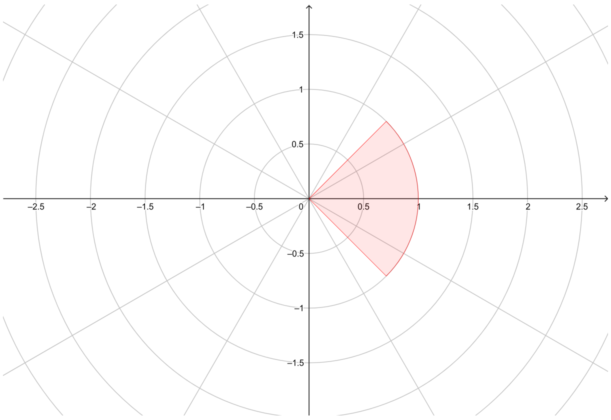

Figure 6: Representação geométrica da curva.

Exercício 2

a)

$$r\sin(\theta) = 0 \Rightarrow y = 0 \text{ (eixo das abscissas)}$$

b)

$$r\cos(\theta) = 0 \Rightarrow x = 0 \text{ (eixo das ordenadas)}$$

c)

$$\begin{aligned} r = \frac{5}{\sin(\theta) - 2\cos(\theta)} &\Rightarrow r \times (\sin(\theta) - 2\cos(\theta)) = 5 \\ &\Rightarrow r\sin(\theta) - 2r\cos(\theta) = 5 \\ &\Rightarrow y - 2x = 5 \Rightarrow y = 5 + 2x \end{aligned}$$

d)

$$\begin{aligned} r = 8\sin(\theta) &\Rightarrow r^2 = 8r\sin(\theta) \\ &\Rightarrow (x^2) + (y^2) = 8y \\ &\Rightarrow x^2 = 8y - y^2 \end{aligned}$$

e)

$$r^2 + 2r^2 \cos(\theta)\sin(\theta) = 1 \Rightarrow r^2 + 2r\cos(\theta) r\sin(\theta) = 1$$
$$\Rightarrow x^2 + y^2 + 2xy = 1$$

f)

$$r = 2\cos(\theta) + 2\sin(\theta) \Rightarrow r^2 = r \times (2\cos(\theta) + 2\sin(\theta))$$
$$\Rightarrow x^2 + y^2 = 2x + 2y$$

Exercício 3

a)

$$r\cos(\theta) = 7$$

b)

$$(r\cos(\theta), r\cos(\theta))$$

c)

$$y^2 = 4x \Rightarrow \frac{y^2}{x} = 4 \Rightarrow y \times \frac{y}{x} = 4 \Rightarrow r\sin(\theta)\tan(\theta) = 4 \Rightarrow r = \frac{4}{\sin(\theta)\tan(\theta)}$$

d)

$$x - y = 3 \Rightarrow r\cos(\theta) - r\sin(\theta) = 3 \Rightarrow r(\cos(\theta) - \sin(\theta)) = 3 \Rightarrow r = \frac{3}{\cos(\theta) - \sin(\theta)}$$

e)

$$xy = 2 \Rightarrow r\sin(\theta) r\cos(\theta) = 2 \Rightarrow r^2 = \frac{2}{\sin(\theta)\cos(\theta)}$$

f)

$$\begin{aligned}
x^2 + xy + y^2 = 1 &\Rightarrow (r\cos(\theta))^2 + r^2\sin(\theta)\cos(\theta) + (r\sin(\theta))^2 = 1\\
&\Rightarrow r^2\sin^2(\theta) + r^2\cos^2(\theta) + r^2\sin(\theta)\cos(\theta) = 1\\
&\Rightarrow r^2(\sin^2(\theta) + \cos^2(\theta) + \sin(\theta)\cos(\theta)) = 1\\
&\Rightarrow r^2(1 + \sin(\theta)\cos(\theta)) = 1\\
&\Rightarrow r^2 = \frac{1}{1 + \sin(\theta)\cos(\theta)}
\end{aligned}$$

Exercício 4

a)

$$r = s \Leftrightarrow 1 + \cos(\theta) = 1 - \cos(\theta) \Leftrightarrow \cos(\theta) = -\cos(\theta) = 0 \Leftrightarrow \theta = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}.$$

Como consideramos $0 \leq \theta \leq 2\pi$ teremos os pontos de interseção das curvas em coordenadas polares: $(0, \frac{\pi}{2})$, $(1, \frac{\pi}{2})$, $(-1, \frac{\pi}{2})$.

b)

$$\begin{aligned}
r = s &\Leftrightarrow 2\sin(\theta) = 2\sin(2\theta)\\
&\Leftrightarrow 2\sin(\theta) = 4\sin(\theta)\cos(\theta)\\
&\Leftrightarrow \sin(\theta) = 2\sin(\theta)\cos(\theta)\\
&\Leftrightarrow \sin(\theta) - 2\sin(\theta)\cos(\theta) = 0\\
&\Leftrightarrow \sin(\theta)(1 - 2\cos(\theta)) = 0\\
&\Leftrightarrow \sin(\theta) = 0 \ \vee \ 1 - 2\cos(\theta) = 0\\
&\Leftrightarrow \theta = k\pi \ \vee \ \theta = \frac{\pi}{3} + \frac{2k\pi}{3}, \ k \in \mathbb{Z}.
\end{aligned}$$

Como consideramos $0 \leq \theta \leq 2\pi$ os pontos de interseção entre r e s em coordenas polares serão: $(0, 0)$, $(0, \pi)$, $(0, 2\pi)$, $(\sqrt{3}, \frac{\pi}{3})$ e $(\sqrt{3}, \frac{2\pi}{3})$.

c)

$$\begin{aligned} r = s \Leftrightarrow \cos(\theta) &= 1 - \cos(\theta) \\ &\Leftrightarrow = 2\cos(\theta) = 1 \\ &\Leftrightarrow \theta = \arccos(\frac{1}{2}) \\ &\Leftrightarrow \theta = \arccos\left(\frac{1}{2}\right) \vee 2\pi - \theta = \arccos\left(\frac{1}{2}\right) \\ &\Leftrightarrow \theta = \frac{\pi}{3} + 2k\pi \vee \theta = \frac{5\pi}{3} + 2k\pi,\ k \in \mathbb{Z}. \end{aligned}$$

Como consideramos $0 \leq \theta \leq 2\pi$ os pontos de interseção entre r e s em coordenas polares serão: $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $(0,0)$ e $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$.

d)

$$s^2 = 2\sin(2\theta) \Rightarrow s = \pm\sqrt{2\sin(2\theta)}$$

Tomemos $s_- = -\sqrt{2\sin(2\theta)}$ e $s^+ = \sqrt{2\sin(2\theta)}$ e analisemos caso à caso:

s^+

$$\begin{aligned} r = s^+ \Leftrightarrow 1 = \sqrt{2\sin(2\theta)} &\Leftrightarrow \sin(2\theta) = \frac{1}{2} \\ &\Leftrightarrow 2\theta = \arcsin\left(\frac{1}{2}\right) \vee \pi - 2\theta = \arcsin\left(\frac{1}{2}\right) \\ &\Leftrightarrow 2\theta = \frac{\pi}{6} \vee \pi - 2\theta = \frac{\pi}{6} \\ &\Leftrightarrow \theta = \frac{\pi}{12} + k\pi \vee \theta = \frac{5\pi}{12} + k\pi,\ k \in .\mathbb{Z}. \end{aligned}$$

s_-

$$1 = -\sqrt{\sin(2\theta)} \Leftrightarrow \sqrt{2\sin(2\theta)} = -1 \Rightarrow \nexists\, \theta \in [0, 2\pi] :\ r = s_-.$$

Assim, como consideramos $0 \leq \theta \leq 2\pi$ os pontos de interseção entre r e s em coordenas polares serão: $\left(1, \frac{\pi}{12}\right)$, $\left(1, \frac{5\pi}{12}\right)$, $\left(1, \frac{13\pi}{12}\right)$ e $\left(1, \frac{17\pi}{12}\right)$.

Exercício 5

a)

Seja α o coeficiente angular, então:

$$r = \sin(\theta) - 1 \Rightarrow \begin{cases} x = r\cos(\theta) = (\sin(\theta) - 1)(\cos(\theta)) = \cos(\theta)\sin(\theta) - \cos(\theta) \\ y = r\sin(\theta) = (\sin(\theta) - 1)(\sin(\theta)) = \sin^2(\theta) - \sin(\theta) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{d\theta} = \cos^2(\theta) - \sin^2(\theta) + \sin(x) \\ \dfrac{dy}{d\theta} = 2\sin(\theta)\cos(\theta) - \cos(\theta) \end{cases}$$

$$\theta = 0 \Rightarrow \begin{cases} \left.\dfrac{dx}{d\theta}\right|_{\theta=0} = 1 \\ \left.\dfrac{dy}{d\theta}\right|_{\theta=0} = -1 \end{cases}$$

$$\Rightarrow \alpha = \frac{-1}{1} = -1$$

b)

Seja α o coeficiente angular, então:

$$r = \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \Rightarrow \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$$\Rightarrow \begin{cases} x = (\cos^2(\theta) - \sin^2(\theta))(\cos(\theta)) \\ y = (\cos^2(\theta) - \sin^2(\theta))(\sin(\theta)) \end{cases}$$

$$\Rightarrow \begin{cases} x = \cos^3(\theta) - \cos(\theta)\sin^2(\theta) \\ y = \cos^2(\theta)\sin(\theta) - \sin^3(\theta) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{d\theta} = -\sin(\theta)\cos(2\theta) - 2\cos(\theta)\sin(2\theta) \\ \dfrac{dy}{d\theta} = \cos(\theta)\cos(2\theta) - 2\sin(\theta)\sin(2\theta) \end{cases}$$

$$\theta = \frac{\pi}{2} \Rightarrow \begin{cases} \left.\dfrac{dx}{d\theta}\right|_{\theta=\frac{\pi}{2}} = 1 \\ \left.\dfrac{dy}{d\theta}\right|_{\theta=\frac{\pi}{2}} = 0 \end{cases}$$

$$\Rightarrow \alpha = \frac{0}{1} = 0$$

c)

O ponto $(2, 0)$ também pode ser escrito em coordenadas polares como $(2, 0)$.Seja α o coeficiente angular, então:

$$
\begin{aligned}
r = 2 - 3\sin(\theta) &\Rightarrow \begin{cases} x = r\cos(\theta) = (2 - 3\sin(\theta))(\cos(\theta)) \\ y = r\sin(\theta) = (2 - 3\sin(\theta))(\sin(\theta)) \end{cases} \\
&\Rightarrow \begin{cases} x = 2\cos(\theta) - 3\cos(\theta)\sin(\theta) \\ y = 2\sin(\theta) - 3\sin^2(\theta) \end{cases} \\
&\Rightarrow \begin{cases} \dfrac{dx}{d\theta} = -2\sin(\theta) - 3\cos(2\theta) \\ \dfrac{dy}{d\theta} = 2\cos(\theta) - 3\sin(2\theta) \end{cases} \\
\theta = 0 &\Rightarrow \begin{cases} \left.\dfrac{dx}{d\theta}\right|_{\theta=0} = -3 \\ \left.\dfrac{dy}{d\theta}\right|_{\theta=0} = 2 \end{cases} \\
&\Rightarrow \alpha = -\frac{2}{3}
\end{aligned}
$$

Exercício 6

a)

$$
\begin{aligned}
r(\theta) = 4 + 2\cos(\theta) \Rightarrow A &= \int_0^{2\pi} \frac{1}{2}(4 + 2\cos(\theta))^2 \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 16 + 16\cos(\theta) + 4\cos^2(\theta) \, d\theta \\
&= \frac{1}{2}\left[18\theta + 16\cos(\theta) + \sin(2\theta)\right]_0^{2\pi} \\
&= 18\pi
\end{aligned}
$$

b)

$$
\begin{aligned}
r(\theta) = a(1+\cos(\theta)) \Rightarrow A &= \int_0^{2\pi} \frac{1}{2}(a + a\cos(\theta))^2 \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} a^2 + 2a^2\cos(\theta) + a^2\cos^2(\theta) \, d\theta \\
&= \frac{a^2}{2}\int_0^{2\pi} 1 + 2\cos(\theta) + \cos^2(\theta) \, d\theta \\
&= \frac{a^2}{2}\left[\frac{3\theta}{2} + 2\sin(\theta) + \frac{\sin(2\theta)}{4}\right]_0^{2\pi} \\
&= \frac{3\pi a^2}{2}
\end{aligned}
$$

Exercício 7

a)

$$
\begin{aligned}
r(\theta) = \theta^2 \Rightarrow \frac{dr}{d\theta} = 2\theta \Rightarrow l &= \int_0^{\sqrt{5}} \sqrt{(\theta^2)^2 + (2\theta)^2} \, d\theta \\
&= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2} \, d\theta \\
&= |_{u^2=\theta^2+4} \int_{\sqrt{4}}^{\sqrt{9}} u^2 \, du \\
&= \left[\frac{u^3}{3}\right]_{\sqrt{4}}^{\sqrt{9}} \\
&= 3\sqrt{9} - \frac{4\sqrt{4}}{3}
\end{aligned}
$$

b)

$$
\begin{aligned}
r(\theta) = \frac{e^\theta}{\sqrt{2}} \Rightarrow \frac{dr}{d\theta} = \frac{e^\theta}{\sqrt{2}} \Rightarrow l &= \int_0^{\pi} \sqrt{\left(\frac{e^\theta}{\sqrt{2}}\right)^2 + \left(\frac{e^\theta}{\sqrt{2}}\right)^2} \, d\theta \\
&= \int_0^{\pi} \sqrt{\frac{e^{2\theta}}{2} + \frac{e^{2\theta}}{2}} \, d\theta \\
&= \int_0^{\pi} \sqrt{e^{2\theta}} \, d\theta \\
&= \int_0^{\pi} e^\theta \, d\theta \\
&= e^\pi - 1
\end{aligned}
$$

c)

$$
\begin{aligned}
r(\theta) = 1 + \cos(\theta) \Rightarrow \frac{dr}{d\theta} = -\sin(\theta) \Rightarrow l &= \int_0^{2\pi} \sqrt{(1+\cos(\theta))^2 + (-\sin(\theta))^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2cos(\theta)}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2(1 + cos(\theta))}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4\left(\frac{1 + cos(\theta)}{2}\right)}\, d\theta \\
&= \int_0^{2\pi} 2\sqrt{\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \\
&= 2\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta \\
&= |_{u=\frac{\theta}{2}}\ \ 4\int_0^{\pi} |\cos(u)|\, du \\
&= 8\int_0^{\frac{\pi}{2}} \cos(u)\, du \\
&= 8
\end{aligned}
$$

d)

$$
\begin{aligned}
r(\theta) = \frac{2}{1-\cos(\theta)} \Rightarrow \frac{dr}{d\theta} = 2\frac{dr}{d\theta}\left(\frac{1}{1-\cos(\theta)}\right) = -\frac{2\sin(\theta)}{(1-\cos(\theta))^2} \\
\Rightarrow l &= \int_{\frac{\pi}{2}}^{\pi} \sqrt{\left(\frac{2}{1-\cos(\theta)}\right)^2 + \left(-\frac{2\sin(\theta)}{(1-\cos(\theta))^2}\right)^2}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \sqrt{\frac{4}{(1-\cos(\theta))^2} + \frac{4\sin^2(\theta)}{(1-\cos(\theta))^4}}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \sqrt{\frac{4(1-\cos(\theta))^2 + 4\sin^2(\theta)}{(1-\cos(\theta))^4}}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \sqrt{\frac{4(1-2\cos(\theta) + cos^2(\theta) + \sin^2(\theta))}{(1-\cos(\theta))^4}}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \sqrt{\frac{8-8\cos(\theta)}{(1-\cos(\theta))^4}}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \sqrt{\frac{8(1-\cos(\theta))}{(1-\cos(\theta))^4}}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \frac{\sqrt{8}}{\sqrt{(1-\cos(\theta))^3}}\, d\theta \\
&= \sqrt{8}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sqrt{(1-\cos(\theta))^3}}\, d\theta \\
&= \sqrt{8}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sqrt{\left(2\sin^2\left(\frac{\theta}{2}\right)\right)^3}}\, d\theta \\
&= \sqrt{8}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2\sqrt{2}\sin^3\left(\frac{\theta}{2}\right)}\, d\theta \\
&= \sqrt{8}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sqrt{8}\sin^3\left(\frac{\theta}{2}\right)}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sin^3\left(\frac{\theta}{2}\right)}\, d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \csc^3\left(\frac{\theta}{2}\right)\, d\theta \\
&= |_{u=\frac{\theta}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^3(u) du \\
&= -\ln(\sqrt{2}-1) + \sqrt{2}
\end{aligned}
$$

Observe que $\int \csc^3(\theta)\, d\theta$ pode ser calculada por partes como se segue:

$$\int u\, dv = uv - \int v\, du$$
$$\int \csc^3(\theta)\, d\theta = \int \csc(\theta)\csc^2(\theta)\, d\theta$$

$$\begin{cases} u = \csc(\theta) \Rightarrow du = -\csc(\theta)\cot(\theta)\, d\theta \\ v = \int \csc^2(\theta)\, d\theta \Rightarrow dv = \csc^2(\theta)\, d\theta \end{cases}$$

$$\begin{aligned}
\int \csc^3(\theta)\, d\theta &= -\csc(\theta)\cot(\theta) - \int -\cot(\theta)(-\csc(\theta)\cot(\theta)\, d\theta \\
&= -\csc(\theta)\cot(\theta) - \int \cot^2(\theta)\csc(\theta)\, d\theta \\
\cot^2(\theta) = \csc^2(\theta) - 1 &\Rightarrow -\csc(\theta)\cot(\theta) - \int (\csc^2(\theta) - 1)\csc(\theta)\, d\theta \\
&= -\csc(\theta)\cot(\theta) - \int \csc^3(\theta)\, d\theta + \int \csc(\theta)\, d\theta \\
&= -\csc(\theta)\cot(\theta) - \int \csc^3(\theta)\, d\theta + \ln|\csc(\theta) - \cot(\theta)| \\
&\Rightarrow 2 \times \int \csc^3(\theta)\, d\theta = -\csc(\theta)\cot(\theta) + \ln|\csc(\theta) - \cot(\theta)| \\
&\Rightarrow \int \csc^3(\theta)\, d\theta = -\frac{\csc(\theta)\cot(\theta)}{2} + \frac{\ln|\csc(\theta) - \cot(\theta)|}{2} + C, C \in \mathbb{R}
\end{aligned}$$