# The *e*-value and the FBST:
## Logical Properties and Philosophical Consequences*


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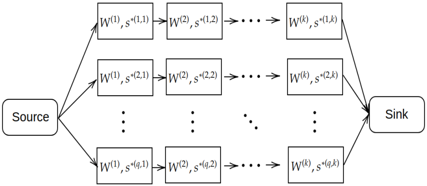


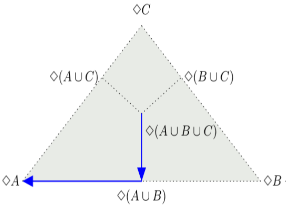


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43rd Bayesian Inference & Maximum Entropy Methods, Ghent, 1-5/07/24

# This Presentation

**(I)** Introduction

**(II)** *e*-value - Definition as statistical significance measure

**(III)** *e*-value - Logic or Compositional calculus for truth values

**(IV)** *Generalized Full Bayesian Significance Test* (FBST)
- Comply with rules of good reasoning from *Modal Logic*

**(V)** Applications to Science and Technology

**(VI)** *e*-value + FBST - Philosophical Consequences:
- *Objective Cognitive Constructivism* epistemological framework
- where *Ontological objects are tokens for eigen-solutions*,
- supporting *Metaphysical* (latent & explanatory) causal laws
(literal: latent= non-observables & gnosiological senses)

**(VII)** Future Research with Logic & Philosophical overlaps

**(VIII)** References, Acknowledgments

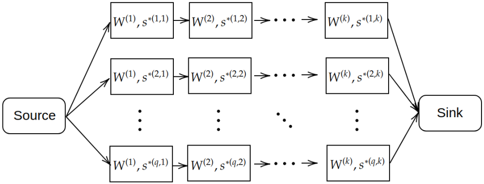
**(FAQs)** Frequently asked questions

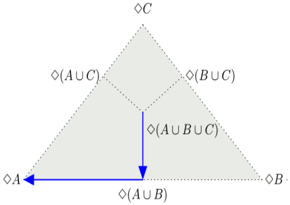
# ev $(H \mid X)$ - *e*-value - epistemic value of $H$ given $X$

- *e*-value, a.k.a. the *epistemic value* of hypothesis $H$ given the observational data $X$ or, the other way around, the *evidence value* rendered by the observational data $X$ in support of hypothesis $H$, is a *significance measure* or *truth value* conceived for statistical modeling by Pereira and Stern (1999).
- (G)FBST - (Generalized) Full Bayesian Significance Test
- Decisions: $\Box H$, accept; $\neg\Diamond H$, reject; or $\nabla H$, remain agnostic.
- Properties related to mathematical statistics and modeling:
- Comply with best principles of Bayesian inference, including *Likelihood princ.*; *Invariance* $(\mathcal{X}, \Theta, H)$; asymptotic *Consistency*.
- Effortlessly accommodates either slack or *sharp* (precise) $H$.
- Excellent modeling properties / operational characteristics:

- Robust and reliable behavior;
- Straightforward formulation & simple numerical implementn;
- FBST outperforms alternatives in several benchmark tests.

# *e*-value and FBST - Logical properties




- a *Logical formalism* can be conceived as an algebra for obtaining truth-values of complex statements from its constituent or elementary parts. In this perspective, the
- *e*-value constitutes a *Probability based Possibilistic calculus*.
- Often, alternatives (*p*-values, Bayes factors) have *no* logic.

- FBST decisions follow rules of modal logic that are natural and intuitive for human interpretation.
- Violating these rules brings the danger of miscommunication, misunderstanding, or misinformation. (alternative tests fail)
- Via mathematical analysis, these rules characterize the FBST.

# *e*-value and FBST - Philosophical consequences

- Distinct significance measures or truth values for statistical hypotheses have distinct logical & operational properties, requir( -ing / -ed ) by distinct epistemological frameworks.

- Thomas Bayes' *Doctrine of Chances* (1763):
- Learn the latent (hiden) *causes* or *natural laws* that describe, predict and explain manifested effects: Equations, = , sharp $H$ !

- Karl Pearson: *New Werther, by Locki* (1880), *Grammar of Science* (1892) and XXth c. *Frequentist statistics* (1896...):
- Radical anti-metaphysical anti-ontological form of positivism;
- Science is only descriptive / predictive, never explanatory;
- Science only reflects or projects the orbserver's *ego*, not the *nature* of an underlying reality (that, if posited, is inaccessible).

- Objective Cognitive Constructivism:
- Scientific knowledge describes / predicts possible interactions of an autopoietic system and objects in its environment, explaining the organization or nature of the interac -tion / -ting parts ⇒
- Naturalized approach to ontology and metaphysics.

# Framework of Parametric Bayesian Statistics

- Given the *sampling* distribution, $p(x \mid \theta)$; *n observations* in the *sample space* $\mathcal{X}$, $X = [x^{(1)}, \ldots x^{(n)}]$; and a *prior* density for the *parameter*(s) $\theta \in \Theta$, $p_0(\theta)$; the *posterior* density for $\theta$, $p_n(\theta \mid X)$, is generated by *Bayesian learning* steps:

$$p_n(\theta) = c_n^{-1}\, p_0(\theta) \prod_{i=1}^{n} p(x^{(i)} \mid \theta) = c_n^{-1}\, p_0(\theta) L(\theta \mid X)$$

- *Hypothesis H* states that $\theta^0$, the true value of the parameter generating $X$, belongs to a region of the parameter space constrained by (vector) inequality and equality constraints,

$$H = \{\theta \in \Theta \mid g(\theta) \leq \mathbf{0} \wedge h(\theta) = \mathbf{0}\}$$

- $h$, the dimension of $H$, is the dimension of the parameter space, $t$, minus the number of equality constraints, $q$, that is, $h = \dim(H) = t - q \leq t = \dim(\Theta)$. (degrees of freedom)
- If $h < t$, $H$ is *sharp* or *precise*; If $h = t$, $H$ is *slack*.

$*$ *Likelihood function*, $L(\theta \mid X)$, is $p(\theta \mid X)$ with free argument $\theta$ and $X$ fixed

# $e$-value definition - Surprise function

**(i)** $s(\theta)$, the *surprise function* of a statistical model, is defined as the quotient between the posterior and the reference densities,

$$s(\theta) = p_n(\theta)/r(\theta)$$

- $r(\theta)$, the *reference density*, can be interpreted as representing vague or weak information about $\theta$, like the uniform density (Laplace), $r(\theta) \propto 1$, an invariant prior (Jeffreys), or a maximum entropy density; see Stern (2011), and Stern & Pereira (2014).

- Alternatively, the reference density can be interpreted as a representation of the parameter space's underlying information metric, $dl^2 = d\theta^t G(\theta) d\theta$, namely, $r(\theta) = \sqrt{\det G(\theta)}$.

- $s(\theta)$ is invariant by a regular reparametrization, $\omega = \phi(\theta)$ :

$$\widetilde{s}(\omega) = \frac{\widetilde{p}_n(\omega)}{\widetilde{r}(\omega)} = \frac{p_n(\phi^{-1}(\omega))\,|J(\omega)|}{r(\phi^{-1}(\omega))\,|J(\omega)|} = s(\theta)\,,\ \ J(\omega) = \left[\frac{\partial\,\theta}{\partial\,\omega}\right] = \left[\frac{\partial\,\phi^{-1}(\omega)}{\partial\,\omega}\right]$$

Invariant/ Min-info. prior: • Harold Jeffreys (1939, 1961). *Theory of Probability*.
• J.Kapur, H.Kesavan (1992). *Entropy Optimization Principles w. Applications*.

# *e*-value definition - Tangential point and set

**(ii)** $s^*$, the maximum (or supremum) of the surprise function constrained to the hypothesis $H$, is defined as

$$s^* = \max_{\theta \in H} s(\theta) \ .$$

A *tangential point* is a maximizing argument, $\theta^* \mid s^* = s(\theta^*)$.

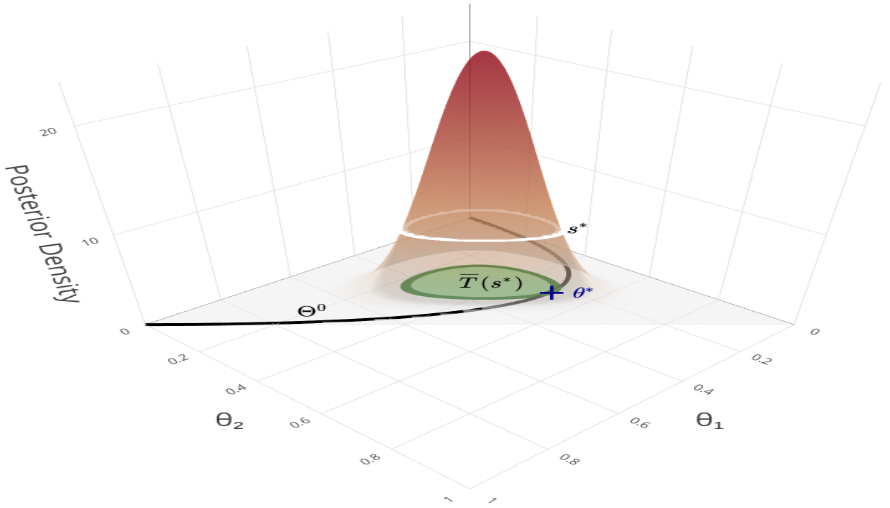
**(iii)** $T(v)$, the closed lower $v$-cut of the surprise function, and its complement, the (open) highest surprise function set (HSFS) at level $v$, $\overline{T}(v)$, are defined as

$$T(v) = \{\theta \in \Theta \mid s(\theta) \leq v\} \ , \quad \overline{T}(v) = \{\theta \in \Theta \mid s(\theta) > v\} \ .$$

The HSFS at level $v = s^*$, $\overline{T}(s^*)$, is called the *tangential set*, for its border corresponds to the projection of the contour line of the surprise function that is tangential to hypothesis $H$.


- David G. Luenberger (1983). *Linear and Nonlinear Programming.*
- Michel Minoux (1986). *Mathematical Programming: Theory and Algorithms.*
- S.C.Fang, J.Rajasekera, H.Tsao (1997). *Entropy Optimization & Math.Prog.*

# *e*-value definition - Visualization of key elements




Pereira, Stern (1999), HWE: $p_n(\theta \mid x) \propto \theta_1^{x_1+y_1-1} \theta_2^{x_2+y_2-1} \theta_3^{x_3+y_3-1}$ ,
$[x_1, x_2, x_3]$= hom/het/hom-zigote counts; $y = \mathbf{1}$, $r \propto 1$, $s(\theta) = p_n(\theta)$,
$\Theta = \{\theta \geq 0 \mid \theta_1 + \theta_2 + \theta_3 = 1\}$ , $H = \{\theta \in \Theta \mid \theta_2 = 2\sqrt{\theta_1}\sqrt{\theta_3}\}$ ,

# $e$-value definition - Truth function and Truth value

**(iv)** $W(v)$, the *truth function* or *Wahrheitsfunktion* at level $v$, is the posterior probability mass inside the lower $v$-cut of the surprise function.

$$W(v) = \int_{T(v)} p_n(\theta)\, d\theta \ ,$$

while its complement is defined as $\overline{W}(v) = 1 - W(v)$.

**(v)** $\text{ev}(H \mid X)$, the *epistemic value* of hypothesis $H$ given the observed data $X$, is defined as the truth function $W(v)$ computed at level $v = s^*$, while its complement, $\overline{\text{ev}}(H \mid X)$, the evidence given by the observed data $X$ against hypothesis $H$, has the complementary probability mass,

$$\text{ev}(H \mid X) = W(s^*) \ , \quad \overline{\text{ev}}(H \mid X) = \overline{W}(s^*) = 1 - \text{ev}(H) \ .$$

- Brian D. Ripley (1987). *Stochastic Simulation*.
- John M. Hammersley, D.C. Handscomb (1964). *Monte Carlo Methods*.
- $h(\theta)$=0 invar.: No $\int_H f(\theta)\, d\mu_H$, only $\int_\Theta f(\theta)\, d\theta$; No nuisance param. elimin.

**(vi)** $\text{sev}(H \mid X)$, the *standardized e-value* of a hypothesis $H$ of dimension $h = \dim(H) \leq t = \dim(\Theta)$, and its complement, $\overline{\text{sev}}(H \mid X)$, are defined as follows:

$$\text{sev}(H \mid X) = 1 - \overline{\text{sev}}(H \mid X)\ ,\quad \overline{\text{sev}}(H \mid X) = \sigma(t, h, \overline{\text{ev}}(H \mid X))$$

where $\sigma(t, h, c)$, the *standardization function* on arguments $t, h \in \mathcal{N}_+$ and $c \in [0, 1]$, is defined in terms of the *chi-square* cumulative distribution with $d$ degrees of freedom, $Q(d, z)$,

$$\sigma(t, h, c) = Q\left(t - h, Q^{-1}(t, c)\right)\ ,\quad Q(k, x) = \frac{\Gamma(k/2, x/2)}{\Gamma(k/2, \infty)}\ .$$

- As the number of observations $n \to \infty$, (+regularity condtns) $\text{sev}(H \mid X)$ exhibits the following asymptotic behavior:

- If $H$ is false, $\text{sev}(H \mid X) \to 0$; If $H$ is true, $\text{sev}(H \mid X) \to U[0, 1]$. Concerning this behavior, $\overline{\text{sev}}(H)$ resembles the classical *p*-value and, accordingly, can replace (and outperform) it in commonly used statistical procedures; Borges & Stern (2007).

- According to Wittgenstein, *Tractatus* (2.0201, 5.0, 5.32):
- Every complex statement can be analyzed from its elementary constituents.
- Truth values of elementary statements are the results of those statements' truth-functions (Wahrheitsfunktionen).
- All truth-function are results of successive applications, to elementary constituents, of a finite number of truth-operations (Wahrheitsoperationen).

- Let us consider alternative elementary hypotheses, $H^{(i,j)}$, $i = 1 \ldots q$, defined in $j = 1 \ldots k$ independent constituent models, $M^{(j)}$, and also a complex hypothesis, $H$, defined by *logical composition* in *homogeneous disjunctive normal form* (disjunction of conjunctions) of the elementary hypotheses in the product model $M = M^{(1)} \times \ldots \times M^{(k)}$, namely,

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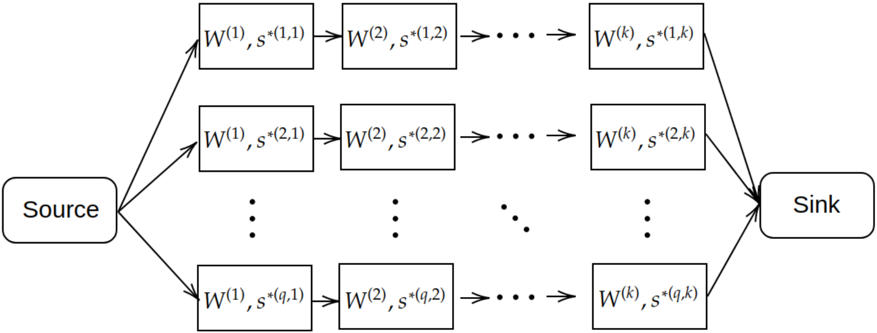
- R.E.Barlow, F.Prochan (1981). *Statistical Theory of Reliability and Life Testing Probability Models*. • A.Kaufmann, D.Grouchko, R.Cruon (1977). *Mathematical Models for the Study of the Reliability of Systems*.




$$H = \bigvee_{i=1}^{q} \bigwedge_{j=1}^{k} H^{(i,j)} \ ,$$

$$M^{(i,j)} = \{\Theta^{(j)}, H^{(i,j)}, p_0^{(j)}, p_n^{(j)}, r^{(j)}\} \ , \quad M = \{\Theta, H, p_0, p_n, r\} \ ,$$

$$\Theta = \prod_{j=1}^{k} \Theta^{(j)}, \ p_0 = \prod_{j=1}^{k} p_0^{(j)}, \ p_n = \prod_{j=1}^{k} p_n^{(j)}, \ r = \prod_{j=1}^{k} r^{(j)}.$$

# *e*-value - Logic or Compositional Calculus

- ev $(H)$, *e*-value of the complex hypothesis, is computed as:

$$\text{ev}\left(\bigvee_{i=1}^{q}\bigwedge_{j=1}^{k} H^{(i,j)}\right) = W\left(\max_{i=1}^{q}\prod_{j=1}^{k} s^{*(i,j)}\right)$$

- where $s^{*(i,j)}$ are elementary maxima, and

- the *truth function* or *Wahrheitsfunktion* of the product model, $W(v)$, is given by the *truth operation* or *Wahrheitsoperation* defined by the Mellin convolution

$$W = \bigotimes_{1\leq j\leq k} W^{(j)} \, , \quad W^{(1)} \otimes W^{(2)}(v) = \int_0^\infty W^{(1)}\left(\frac{v}{y}\right) W^{(2)}(y)\, dy$$


- M.D. Springer (1979). *The Algebra of Random Variables*. Wiley.
- R.C.Williamson (1989). *Probabilistic Arithmetic*. Univ. of Queensland.
- S.Kaplan, C.Lin (1987). An Improved Condensation Procedure in Discrete Probability Distribution Calculations. *Risk Analysis*, 7, 15-19.

# *e*-value - Logic or Compositional Calculus

- The distribution of the product of two independent positive random variables is the Mellin convolution of their distributions.
- Hence, $\otimes$ is a commutative and associative operator.
- Moreover, in the extreme case of null-or-full support, i.e., when, for $1 \leq i \leq q$ and $1 \leq j \leq k$, $s^{*(i,j)} = 0$ or $s^{*(i,j)} = \widehat{s}^{(j)}$, the *e*-values of the constituent elementary hypotheses are either 0 or 1, and the conjunction and disjunction composition rules of classical logic hold.
- No similar compositional calculus, i.e., Logic, exists for most statistical truth values (in statistics, significance values) !
- The *e*-value can be characterized a *possibility measure* obtained by a transformation of the posterior density, see


- A.Y.Darwiche, M.Ginsberg (1992). A Symbolic Generaliz. of Probab. Theory.
- D. Dudois, H. Prade (1982). On Several Representations of an Uncertain Body of Evidence. pp.167-181, Fuzzy Information and Decision Processes.
- G.J. Klir, T.A. Folger (1988). *Fuzzy Sets, Uncertainty and Information.*

# Support structures for some Abstract Belief Calculi

$$a = \Phi(A),\ \ b = \Phi(B),\ \ c = \Phi(C = A \wedge B)$$

| $\Phi(\mathcal{U})$ | $a \oplus b$ | $\mathbf{0}$ | $\mathbf{1}$ | $a \preceq b$ | $c \oslash a$ | $a \otimes b$ | Calculus |
|---|---|---|---|---|---|---|---|
| $[0,1]$ | $a+b$ | $0$ | $1$ | $a \leq b$ | $c/a$ | $a \times b$ | Probability |
| $[0,1]$ | $\max(a,b)$ | $0$ | $1$ | $a \leq b$ | $c/a$ | $a \times b$ | Possibility |
| $\{0,1\}$ | $\max(a,b)$ | $0$ | $1$ | $a \leq b$ | $\min(c,a)$ | $\min(a,b)$ | Classic.Logic |
| $[0,1]$ | $a+b-1$ | $1$ | $0$ | $b \leq a$ | $(c-a)/(1-a)$ | $a+b-ab$ | Improbablty |
| $\{0..\infty\}$ | $\min(a,b)$ | $\infty$ | $0$ | $b \leq a$ | $c-a$ | $a+b$ | Disbelief |

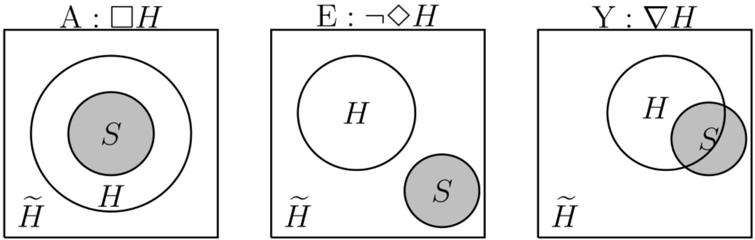
The *support summation*, $\oplus$, operator gives the support value of the disjunction of two logically disjoint statements, i.e.,

$$\neg(A \wedge B) \Rightarrow \Phi(A \vee B) = \Phi(A) \oplus \Phi(B) .$$

The support scaling operator, $\oslash$, gives the conditional support value of $B$ given $A$ from the unconditional support values of $A$ and the conjunction $C = A \wedge B$, i.e.,

$$\Phi_A(B) = \Phi(A \wedge B) \oslash \Phi(A) .$$

# Generalized Full Bayesian Significance Test




Testing $H$ with a region estimator $S$: Accept if $S \subset H$; reject if $S \subset \overline{H}$; and agnostic otherwise = if $H$ intersects both $H$ and $\overline{H}$.
GFBST uses region estimators $S = \overline{T}(v) = \{\theta \in \Theta \,|\, s(\theta) > v\}$

- *Generalized Full Bayesian Significance Test*, GFBST:
- *Rejects H* (impossibility, $\neg\Diamond H$), if its *e*-value stays below an established threshold, $c$, that is, if $\text{ev}\,(H) < c$;
- *Accepts H* (necessity, $\Box H$) if it rejects its complement, that is, if $\text{ev}\,(\overline{H}) < c$, where $\overline{H} = \Theta - H$; and remains
- *Agnostic* otherwise (contingency, $\nabla H = \Diamond H \wedge \neg\Box H$).

# GFBST - Modal Logic rules of good reasoning

- Since the GFBST is directly engendered by the *e*-value, it inherits all its good statistical and compositional properties.

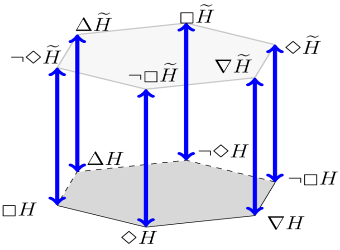
- The GFBST obeys the rules of modal logic for consistent reasoning (concerning $\Diamond H$, $\Box H$, $\nabla H$) presented in the sequel.

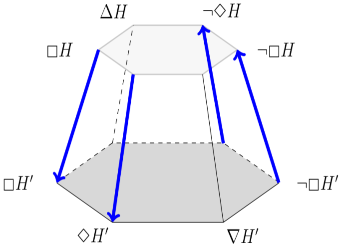
- These rules for consistent reasoning correspond to basic principles of rational argumentation that are natural and intuitive for human interpretation.

- Using inference or decision procedures that violate these rules of good reasoning brings the danger of miscommunication, misunderstanding, or misinformation.

- Using mathematical analysis, these same rules give an (essentially) unique characterization of the GFBST, see Stern (2003), Esteves et al. (2016) and Stern et al. (2018).

- Alternative tests (by *p*-value or Bayes factors) often fail.




(I) *Invertibility:* For any $H$ and its complement, $\overline{H} = \Theta - H$;

(I.i) *Necessity inversion:* $\Box H \Leftrightarrow \neg\Diamond\overline{H}$;

(I.ii) *Possibility inversion:* $\Diamond H \Leftrightarrow \neg\Box\overline{H}$;

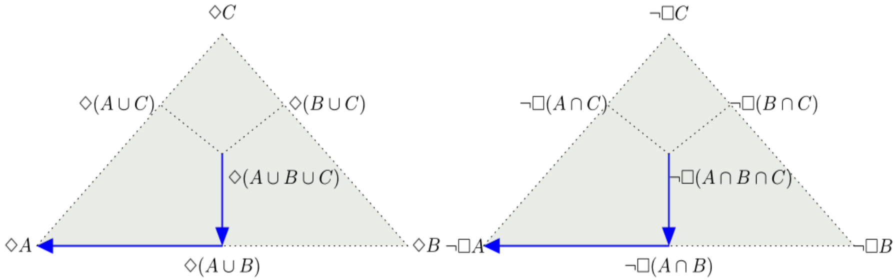
(I.iii) *Contingency inversion:* $\nabla H \Leftrightarrow \nabla\overline{H}$.

(M) *Monotonicity:* For any $H$ and a superset, $H' \supset H$;

(M.i) *Monotonic necessity:* $\Box H \Rightarrow \Box H'$;

(M.ii) *Monotonic possibility:* $\Diamond H \Rightarrow \Diamond H'$.

# GFBST - Consonance rules




(C) *Consonance:* For any indexed set of hypotheses, $H^{(i)}$ , $i \in I$;

(C.i) *Union consonance*: $\Diamond(\cup_{i \in I} H^{(i)}) \Rightarrow \exists i \in I \,|\, \Diamond H^{(i)}$;

(C.ii) *Intersection consonance*: $\forall i \in I,\ \Box H^{(i)} \Rightarrow \Box(\cap_{i \in I} H^{(i)})$.

See Esteves et al. (2016, 2019), Stern et al. (2003, 2004, 2018), and also

- Richard H. Gaskins (1992). *Burdens of Proof in Modern Discourse*.
- Robert Blanché (1953). Sur l'Opposition des Concepts. *Theoria*, 19, 89-130.
- Leonard Nelson (1921). *Typische Denkfehler in der Philosophie*.

# *e*-value - Applications to Science and Technology

• The development of statistical significance measures and tests may be motivated by their intended theoretical properties, which, in turn, may be inspired by epistemological desiderata.

• These significance measures and tests must also prove themselves on the battlefields of technology and science as effective, efficient, robust, and reliable tools for the trade.

• Hundreds of published applications listed in the surveys Pereira & Stern (2020) and Stern et al. (2022).

• Oscar Kempthorne (1978) challenge to Carlos Pereira:
– Define a Bayesian test for sharp statistical hypotheses that works at least as well as frequentist tests based on *p*-values.
• Done! ...also Logical (compositional), Likelihood principle, fully invariant, robust, more powerful, easy to implement, etc.

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• Research in Logic, formal methods & foundations of science - do have - deep and important philosophical consequences!

# Philosophically Consequent XVIII - IXX c. Statistics

- Distinct significance measures or truth values for statistical hypotheses have distinct mathematical & operational properties requir(-ing / -ed by) distinct epistemological frameworks.

- Spinoza's epistemic principles of *Ethica* (1677), Stern (2018):
- *Deus sive natura* (God acts by *Invariant Laws*, causa / cousa)
- *Cognitione causae & Leges naturae universales* ($\Leftrightarrow$ nature)
- *Amor Dei intellectualis* (knowing them is possible and good)

- Rev.* (Thomas) Bayes rule, *Doctrine of Chances* (1763):

– How to learn about causes or latent (hiden) paramaters, $\theta$, from observed (sampled) consequences, $X = [x^{(1)}, \ldots x^{(n)}]$.

- Followed by Pierre-Simon de Laplace (1811). 1st Bayesians' language: *probabilities of causes*, today: *inverse-probability*
- George Boole, *Laws of Thought* (1854), problem X on direct probability, easy, vs. problem IX on inverse probability, hard.

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$\ast$ Nicolaus of Damascus (4.AD), Theodoricus of Chartres (1140), compared: material, efficient, formal, final causes $\sim$ 4 elements, Father, Son, Holy spirit; see Fazzo & Zonta (2008), Aristotle's theory of causes and the holy trinity.

# Philosophically Consequent XXth c. Statistics

- Carl Pearson (1857-1936), starts as a student of Spinoza;
- Karl Pearson: *New Werther by Locki* (1880) Fichtean inverted Spinozism: Science learns about the *ego*, not about the world;
- *Grammar of Science* (1892) and statistical works (1896...) adopt a radical form of positivism, deprecating "metaphysical speculation as to the causes" and promoting "careful collection of sufficient data", seeking only accurate description / prediction;
- Never test if $\theta \in H$ within the framework of a *Scientific Theory*!
- Only test if an *empirical model*, $p(x \mid \theta)$, offers a *good fit* to observational data for a *fixed but unknown parameter*, $\theta^0$!
- K.Pearson's anti -metaphysical / -causal / -explanatory ideas live on in the philosophically-correct language of XX c. statistics.
- XXth c. *Frequentist* school: Ronald Fisher (1912, 1922), etc;

– Fisher alternative *fiducial* statistics (1935) had minor impact;

- Bruno de Finetti (1937) reintroduces probabilistic parameters as subjective and transitory variables for predictive inference;
- de Finetti Bayesians could remain faithful & loyal positivists.

# Objective Cognitive Constructivism & Eigen-Solutions

- *e*-value + FBST + *Objective Cognitive Constructivism* epistemological framework offer a natural / scientific path to approach Ontology and Metaphysics, returning to Bayes'
- *Cognitione causae & Leges naturae universales*

- Fundamental metaphor: In a *Scientific Ontology* (comp. sci.)
- *Objects are tokens for eigen-solutions!* paraphrasing
- H. v.Foerster (2001): *Objects are tokens for eigen-behaviors!*
- Eigen-solutions emerge as invariant entities, that is, as operational eigen-solutions, equilibrium-states, fixed-points, invariant-forms, standing-modes, constant-behaviors, etc. for an autonomous system interacting with its environment.
- Objects, and the names or words we use to call (label) them, stand for and point at such invariant entities.
- Words can be articulated in language (comp. sci. ontology), whose grammar and semantics, should, somehow, correspond to the composition rules for the objects these words stand for.

# Eigen -Values / -Vectors / -Solutions  Metaphor

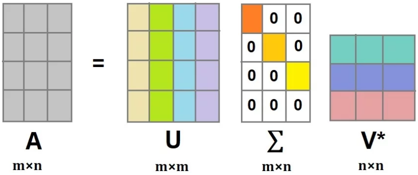
- *4 essential properties* of (non-degenerate) Eigen-Solutions:

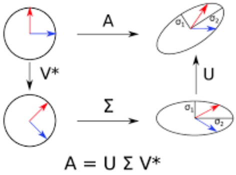
– example: SVD or sigular value decomposition of A;

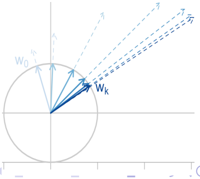
- *Discrete* set of *Precisely* defined singular values & vectors that are invariant objects of a given linear operator or matrix A;
- There are (numeraically) *Stable* (algorithmically feasible) ways to compute / obtain these eigen-solutions;

– ex: Power meth. for largest eigenvector: $v_{k+1} := (1/||v_k||)Mv_k$

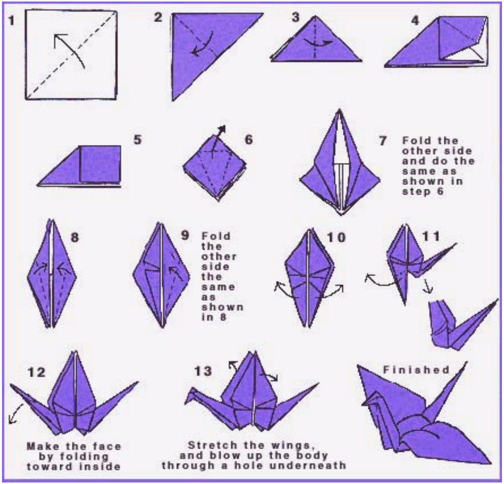
- Orthonormal (unitary) matrices U and V are *bases* for the domain and image spaces of matrix A,  i.e., any vector can be *de-Composed* (separated, by projection) and *re-Composed* (by superposition) from such invariant elements (of $U$, $\Sigma$, $V$ );

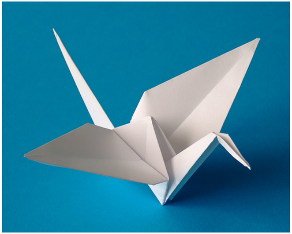


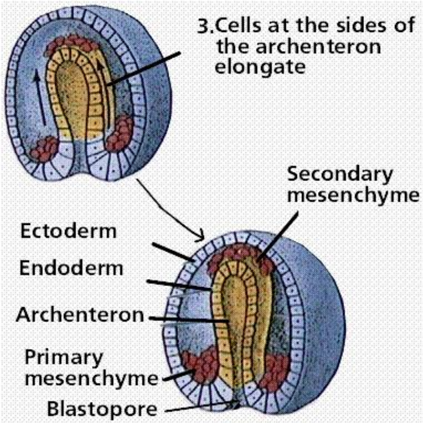
# Eigen-Solutions & R. Dawkins' Origami Metaphor

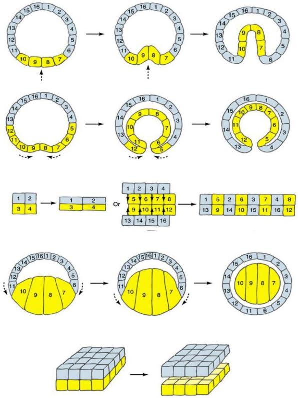


Crane (Tsuru) origami instructions using a *basis* of folding operations*.
What happens when we play Chinese whispers game with both cranes?
Why? ∗ 4 essential properties: *Exact*, *Stable*, *Separable* & *Composable* !

# Eigen-Solutions - Generalized Origami Metaphor



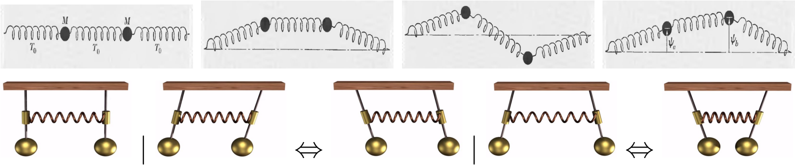



Real (biologic) cranes are self-assembled using the tissue folding *basis* of organic morphogenesis: (left) Gastrulation / (right) foldings:
- Invagination, involution, convergent extension, epiboly, delamination.

- Operations' 4 essential properties based on underlying symmetries!

# Dynamic Invariants* & Eigen-Vectors (discrete)

*Bases* for two coupled oscillators: Transverse and longitudinal

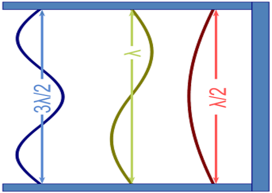


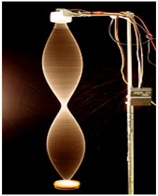
Left: Static invariant states (equilibrium) for the two systems.
Right: Dynamic invariant states for these systems:
Two *normal modes* of movement for the oscillating particles:
Symmetric mode – same amplitude and same phase,
Antisymmetric mode – same amplitude but opposite phases.

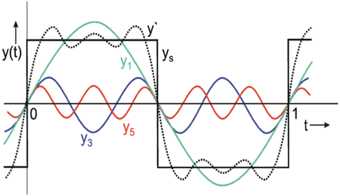
- (De)Composition: Any free movement of these systems is a linear superposition of their normal modes (eigen-solutions).
- Stability: Energy stored at each normal mode is constant.
- Precision: System's Symmetries impose strict invariant (eigen) forms and oscillating factor frequencies (eigen-values)

∗ Eigen-forms* ! see Stern (2008); J.N.Franklin (1968) *Matrix Theory*.

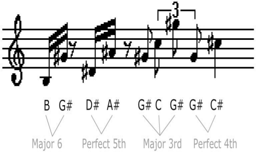
# Dynamic Invariants & Eigen-Functions (continuum)

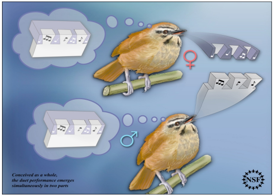





- Continuous string: $\lambda_n = 2L/n$, harmonic freq. $k_n = 2\pi/\lambda_n$.


- Grammar 1: Musical scales & harmonic chords (men, wrens)
- Perceived & used by essential properties of eigen-solutions;
- Eigen-Solutions (objects & relations) can be named! (men)
- Grammar 2: Linear Algebra and Differential Calculus (later)

# Objective Cognitive Constructivism & Eigen-Solutions

- The *4 essential properties* of eigen-solution type entities:
- *Precise*, hence represented by sharp $H \subset \Theta$, for, in exact sciences, natural laws are expressed as equations; $=$
- *Stable*, corresponding to measurable manifested observables;
- *Separable* and *Composable*, so can be individually accessed & rationally combined for building / analyzing complex systems.

- Scientific laws are interpreted within frameworks including metaphysical theories, experimental means and methods, etc.

- Verifying / validating these exact laws provides ontological grounding or empirical **anchor** points to scientific theories.

- The ontological status of an object in a scientific discipline – including its associated metaphysical (latent / explanatory) concepts in a theory +technological means and methods – corresponds to how well it manifests the 4 essential properties of an eigen-solution in cyclic processes of scientific production; see Stern (2011, 2017, 2020).

# Precise (=) Invariant Laws & Eigen-Solutions

- Newton laws: (4) Gravitation: $f = \gamma\ \mu_1 \mu_2 / r^2$ ,

$=$ $=$

\- $\gamma$ value / precision: $\gamma = 6.67430(15) \times 10^{-11}$ ;
(1) Inertial motion; (3) Interaction by opposing forces;
(2) Forces cause (vector) accelerations, $d^2 x / dt^2 = f / \mu$ .

- Latin observables, $x$ (data); Greek parameters, $\theta$ (estimated).

(a) Complex systems analyzed by (de/re)composition +4 laws.
(b) Newtonian mechanics, measurement devices, calculation methods, etc. are developed and perfected in a process that, w. increasing precision, validates (or not) Newton (exact) laws.

- For analogies between *Invariant Laws* and *Eigen-Solutions*; and their *statistical verification* for *ontological grounding*, see Stern (2011, 2017, 2020) and references therein. Hints:

(a) A symmetric matrix $M$ has an *orthogonal basis* (decoupling coord.system) of (scaled invariant) eigen-vectors, $v_k = \lambda_k^{-1} M v_k$ .
(b) A regular Markov chain kernel $M$ has a (unique) asymptotic equilibrium distribution or eigen-vector, $v$ , such that $v = vM$

# Recurrent Production of Eigen-Solutions Metaphor

| Theoretical | \| | Metaphysical | \| | Experimental |
|---|---|---|---|---|
| Mathematical formalization | $\Rightarrow$ | Causal explanation | $\Rightarrow$ | Hypotheses formulation |
| $\Uparrow$ | | | | $\Downarrow$ |
| Speculative interpretation | | Eigen-solution verification ⚓ | | Experiment (trial) design |
| $\Uparrow$ | | | | $\Downarrow$ |
| Statistical modeling | $\Leftarrow$ | Data acquisition | $\Leftarrow$ | Technological implementation |
| Parameter spc; $\Theta$ | \| | Operational | \| | Sample space; $\mathcal{X}$ |

Science cyclic production (auto-poiesis) matrix; Stern (2017, 2020)

⚓ *e*-value + FBST are tailor-made tools for evaluating / validating the *4 essential properties* of eigen-solutions, namely, being:
⚓ *precise*, *stable*, *separable*, and *composable*.

- $v_{k+1} := (1/||v_k||)Mv_k \xrightarrow{\text{conv}} \sim$ laws, const, methds.. $\in$ ontology
- Hard science criteria ⚓ $\Rightarrow$ No wishful (bad circular) thinking!

Heinz von Foerster in Segal (2001, p.127, 145, 266):

- *Objects are tokens for eigen-behaviors [eigen-solutions]. This is the constructivist's insight into what takes place when we talk about our experience with objects.*

- *Eigenvalues have been found ontologically to be discrete [exact, precise, sharp], stable, separable and composable, while ontogenetically to arise as equilibria that determine themselves through circular processes. Ontologically, Eigenvalues and objects, and likewise, ontogenetically, stable behavior and the manifestation of a subject's "grasp" of an object, cannot be distinguished.*

- *The meaning of recursion is to run through one's own path again. Under certain conditions there exist indeed solutions which, when reentered into the formalism, produce again the same solution.* • They are called eigen- equilibrium- invariant- standing- fixed-... -solution -state -behavior -mode -point , etc.

Humberto Maturana & Francisco Varela (1980, p.10, 78-79, 84):

- *Autopoietic systems are organized (defined as a unity) as a network of processes of production (transform., destruction) of components that, through their interactions and transformations, continuously regenerate and realize the same network*.

- *This circular organization implies predictions: Interactions that took place once will take place again... Every interaction is a particular interaction, but every prediction is a prediction of a class of interactions. This makes living systems inferential systems, and their domain of interactions a cognitive domain*.

✗ ex1: Virus (RNA), active replicating agent; parasite, *not-alive*.
✓ ex2: Bacterium (DNA), strange-loop that recursively renews its structure & components during its lifetime, Bertalanffy (1969)

- *A niche is defined by the classes of interactions into which an organism can enter... If [...a] system predicts a niche that cannot be actualized, it disintegrates.*

- ex3: A minor RNA or DNA mutation (imprecision) can be fatal.

# *e*-value & FBST - Future research w. Logic overlaps

- Statistics in (Un)Countable Sentential Probability
- Long standing tradition of finitary sentential probability;

- Appealing: Theoretical simplicity, computational efficiency, etc;
- Limitation: Can't express measure-theoretic arguments of math. statistics; ex: Modal logic characterization of GFBST;

- Solution(s)? Infinitary second order or fixed-point logics? Explore model's topology? Compact, 1st / 2nd -Countable? Explore convergent e-values and GFBST test procedures by progressive refinement of nested finite probability models?

- Functional Compositionality Structures
- *e*-value Compositional Calculus has a structure similar to possibilistic Abstract Belief Calculus (Darwiche + Ginsberg);
- Similar to statistical *reliability theory* for analysis of complex systems assembled by serial / parallel composition of elements;
- Similar to complex systems' *sensitivity analysis* methods;
- Rigorous unifying framework for similar concepts+methods ?
- Category theory? Alternative formal abstraction theories?

# *e*-value & FBST - Future research w. Logic overlaps

- Rough and Fuzzy Sets

• In the GFBST framework, a sharp hypothesis can be either rejected or remain contingent, but can never be accepted :-(
• How to enlarge an underlying sharp hypothesis into a slack *pragmatic hypothesis*? see Esteves et. al. (2019)
• Objective Cognitive Constructivism $\Rightarrow$ *sharp H* $\Rightarrow$ *Crisp H*;
• How to best incorporate metrological error bounds, methodological imprecisions, fundamental constant uncertainties, etc.?
• Good *Rough* or *Fuzzy* set representations for pragmatic *H*?

- Law, Complexity, and (In)Consequence in Social Systems

• *Natural law* metaphor transported to early modern science strong normative character of social law over human behavior;
– Time to travel this metaphoricl path in the opposite direction?
– Can lessons learned in science benefit social systems?
– Niklas Luhmann legal systems' theory: Congruent (convergent) generalization of normative behavioral expectations.
• How to measure / control (hyper) complexity in legal systems?

# Constructivist Epistemologies - Future Research

- There is a well-established tradition of considering, as autopoietic systems, abstractions of living beings like bee-hives, social systems, cellular automata, genetic systems, etc.
- There are also objections to these abstractions of autopoiesis.
- The best ideas quickly outgrow their original scope, and soon find their way far beyond their first intended applications.
- Autopoiesis metaphor can bridge across distinct applications, transporting, extending and generalizing valuable knowledge.
- Quantitative tools like the *e*-value +FBST can be used to evaluate or select good models for such applications.


- N.Luhmann (1989). *Ecological Communication*. Chicago Univ. Press.
- B.McMullin, F.Varela (1997). Rediscovering Computational Autopoiesis.
- F.Varela, H.Maturana, R.Uribe (1974). Autopoiesis: The Organization of Living Systems. Its Characterization and a Model. *BioSystems*, 5, 187-196.
- M.Zeleny (1980). *Autopoiesis, Dissipative Struct., Spont. Social Orders*.
- Milan Zeleny (1981). *Autopoiesis: A Theory of Living Organization*.
- R.Inhasz, J.M.Stern (2010). Emergent Semiotics in Genetic Programming & the Self-Adaptive Semantic Crossover. Stud. Comput. Intell., 314, 381-392.

# Hubris of Radical Constructivism & Postmodernism

- Radical Constructivism of late H. Maturana (1991):

*- Scientific explanations arise operationally as generative mechanisms accepted by us as scientists through operations that do not entail or imply any supposition about an independent reality, so that in fact there is no confrontation with one, nor is it necessary to have one...*
*- Quantification (or measurements) and predictions can be used in the generation of a scientific explanation but do not constitute the source of its validity. The notions of falsifiability (Popper), verificability, or confirmation would apply to the validation of scientific knowledge only if this were a cognitive domain that revealed, directly or indirectly, by denotation or connotation, a transcendental reality independent of what the observer does...*
*- Nature is an explanatory proposition of our experience with elements of our experience. Indeed, we human beings constitute nature with our explaining, and with our scientific explaining we constitute nature as the domain in which we exist as human beings (or languaging living systems).*


- J.M. Stern (2007). Language & Self-Reference Paradox. *C&HK*, 14, 71-92.
- H.R. Maturana (I), F.J. Varela (1980). *Autopoiesis and Cognition.* -Biology
- Heinz von Foerster (2001), *The Dream of Reality*. (2003), *Understanding*[2].
- W. Rasch (2000). *Niklas Luhmann's modernity: Paradoxes of differentiation.*
- H. Maturana (II) (1991, p.36-44). Science & Reality in Daily Life. -Psychlgy
- J.Efran, M.Lukens, R.Lukens (1990). *Language, Structure and Change.*

- Develop and calibrate standardization transforms for hypotheses at the border of parametric space, $\partial\Theta = \Theta - \Theta^\circ$, or for non-regular hypotheses, like discontinuous or self-intercepting algebraic sub-manifolds

- Theoretical references:

• L.Ventura et al. (2015, 2020) on higher order asymptotic approximations
• H. Chernoff (1954). On the Distribution of the Likelihood Ratio.
• S.G. Self, K.Y. Liang (1987). Asymptotic Properties of Maximum Likelihood Estimators and Likelihood Ratio Tests Under Nonstandard Conditions.
• M. Drton (2007). Likelihood ratio tests and singularities.
• D.W.K. Andrews (2001). Testing When a Parameter is on the Boundary of the Maintained Hypothesis.

- Some possible Applications:

• M.S. Lauretto et al. (2005). FBST for Mixture Model Selection.
" (2007). The Problem of Separate Hypotheses via Mixture Models.
• A. Schwartzman, W.F. Mascarenhas, J.E. Taylor (2009). Inference for eigenvalues and eigenvectors of Gaussian symmetric matrices

- doi:10.2307/2692206 doi:10.1214/07-AOS571 doi:10.2307/2289471 doi:10.1063/1.2821272 doi:10.1214/08-AOS628

- Develop Non-parametric, Semi-parametric or Infinite parameter models in amenable frameworks for *e*-value and FBST Bayesian inductive statistical inference. ex:

- Truncated expansions in Fourier, Orthogonal polynomials, Wavelets, and other infinite bases for functional spaces;

- Incomplete matrix factorization methods.

– Development of interpretable informative priors expressing constitutive conditions; ex: decay rates for series coefficients in solutions having finite energy, smoothness conditions, etc.


- References: • J.M. Stern (2020). A Sharper Image: The Quest of Science and Recursive Production of Objective Realities, Sec.6
`doi:10.5007/1808-1711.2020v24n2p255`
- L.A. Sadun (2007). Applied Linear Algebra: The Decoupling Principle. AMS.
- S.I. Tomonaga (1962). Quantum Mechanics, Vol.I. North-Holland. (history of avoiding / controlling ultraviolet catastrophe in black body radiation)
- Fernando Poliano Tarouco Corrêa Flh. (2024). Testes não paramétricos para árvores de Pólya: Versões não paramétricas do FBST.

# *e*-value and FBST - Math. Statistics research topics

- Development of efficient computational methods based on specially adapted Monte Carlo procedures; (associated with)
- Asymptotic expansions of the e-value;
- Integrated optimization procedures (good initial points);
- Advanced condensation and convolution procedures.

- References: • Good Python / R / Matlab FBST pack. needed!


• C.J.P. Bélisle, H.E. Romeijn, R.L. Smith (1993). Hit-and-run algorithms for generating multivariate distributions. `doi:10.1287/moor.18.2.255`

• R. Karawatzki, J. Leydold, K. Pötzelberger (2005). Automatic Markov Chain Monte Carlo Procedures for Sampling from Multivariate Distributions. `epub.wu.ac.at/id/eprint/1400`

• Z.B. Zabinsky, R.L. Smith (2013). Hit-and-Run Methods. pp. 721-729 in: S.I. Gass, M.C. Fu. Encycl. Operations Research and Management Science.

• S. Cabras, W. Racugno, L. Ventura (2015). Higher order asymptotic computation of Bayesian significance tests for precise null hypotheses in the presence of nuisance parameters. *J. Stat. Comput. Simul.*, 85, 2989-3001.

• E. Ruli, N. Sartori, L. Ventura (2020). Robust approximate Bayesian inference. *J. Stat. Plan. Inference.* 205, 10-22.

• S. Kaplan, J.C. Lin (1987). An Improved Condensation Procedure in Discrete Probability Distribution Calculations. *Risk Analysis*, 7, 15-19.

# References - *e*-value and FBST


- W.S. Borges, J.M. Stern (2007). The Rules of Logic Composition for the Bayesian Epistemic E-Values. *Log. J. IGPL*, 15, 5/6, 401-420.
- L.G. Esteves, R. Izbicki, J.M. Stern, R.B. Stern (2016). The Logical Consistency of Simultaneous Agnostic Hypothesis Tests. *Entropy*, 18, 256.
- L.G. Esteves, R. Izbicki, R.B. Stern, J.M. Stern (2019). Pragmatic Hypotheses in the Evolution of Science. *Entropy*, 21, 883, 1-17.
- M.R. Madruga, L.G. Esteves, S. Wechsler (2001). On the Bayesianity of Pereira-Stern Tests. *Test*, 10, 291-299.
- M.R.Madruga, C.A.B.Pereira, J.M.Stern (2003). Bayesian Evidence Test for Precise Hypotheses. *J. Stat. Plan. Inference.* 117, 2, 185-198.
- C.A.B. Pereira, J.M. Stern (1999). Evidence and Credibility: Full Bayesian Significance Test for Precise Hypotheses. *Entropy*, 1, 99-110.
- C.A.B. Pereira, J.M. Stern, S. Wechsler (2008). Can a Significance Test be Genuinely Bayesian? *Bayesian Analysis*, 3, 79-100.
- C.A.B. Pereira, J.M. Stern (2022). The e-value: A Fully Bayesian Significance Measure for Precise Statistical Hypotheses and its Research Program. *São Paulo J. Math. Sci.*, 16, 566-584. doi:10.1007/s40863-020-00171-7
- J.M. Stern (2003). Significance Tests, Belief Calculi, and Burden of Proof in Legal and Scientific Discourse. *Front. Artif. Intell. Appl.*, 101, 139-147.
- J.M. Stern (2004). Paraconsistent Sensitivity Analysis for Bayesian Significance Tests. *LNAI*, 3171, 134-143, 2004.

# References - *e*-value and FBST


- J.M. Stern (2007a). Cognitive Constructivism, Eigen-Solutions, and Sharp Statistical Hypotheses. *Cybernetics & Human Knowing*, 14, 1, 9-36.
- J.M. Stern (2008). Decoupling, Sparsity, Randomization, and Objective Bayesian Inference. *Cybernetics & Human Knowing*, 15, 2, 49-68.
- J.M. Stern (2011a). Symmetry, Invariance and Ontology in Physics and Statistics. *Symmetry*, 3, 611-635.
- J.M. Stern (2011b). Constructive Verification, Empirical Induction, and Falibilist Deduction: A Threefold Contrast. *Information*, 2, 4, 635-650.
- J.M. Stern (2014). Jacob's Ladder and Scientific Ontologies. *Cybernetics and Human Knowing*, 21, 3, 9-43.
- J.M. Stern, C.A.B. Pereira (2014). Bayesian Epistemic Values: Focus on Surprise, Measure Probability! *Log. J. IGPL*, 22, 236-254.
- J.M. Stern (2017). Continuous Versions of Haack's Puzzles: Equilibria, Eigen-States and Ontologies. *Log. J. IGPL*, 25, 4, 604-631.
- J.M. Stern (2018). Karl Pearson on Causes and Inverse Probabilities: Renouncing the Bride, Inverted Spinozism and Goodness-of-Fit. *South American Journal of Logic*, 4, 1, 219-252.
- J.M. Stern, R. Izbicki, L.G. Esteves, R.B. Stern (2018). Logically-Consistent Hypothesis Testing & the Hexagon of Oppositions. *Log. J. IGPL*, 25, 741-757.
- J.M. Stern (2020). A Sharper Image: The Quest of Science and Recursive Production of Objective Realities. *Principia*, 24, 2, 255-297.

# Thank you! Gracias! Hartelijk bedankt! Grato!

IME-USP

Metaphysical theory;
Science & Technology;
Ontological grounding.

- Bug as feature syndrome (common in software industry)
- Killer apps w. top performance benchmarks (I / II error rates):

- Multivariate Normal structure tests, *Braz. J. Prob. Stat.* (2003).
- Reject - (1) Unspecific: Many special cases lumped together; (2) Simplistic: No nuisance parameter elimination technique, No need for (due consideration to) special measure on $\Theta_H$; (3) Unfair: Smart use of (steals work already done on) basic model formulation & numerical methods for integration in $\Theta$, neglecting special work required to deal with each specific $H$. (4) Potential start of a (predatory) publication industry; (5) Need for USA / EU confirmed performance benchmarks.

- Unit root & Cointegration, *AIP* (2007), *Commun. Stat.* (2011) No *ad hoc* priors; Jeffreys (invariant) prior works fine!
- Reject - (1) "Anyone knows" (community competence criterion) that *artificial* priors are required in these cases. (2) Hundreds of published papers developing special priors confirm this (bad-method induced) need (self protection).

# FAQ1: e-value / FBST objections - Theoretical

- First version of the FBST (1999) was Not invariant;

- Reference density and Surprise function published (2003) $s(\theta) = p_n(\theta)/r(\theta)$, to accomplish full and explicit invariance;
- Objection-(1): "Cancel the prior" If (usually not) $p_0(\theta) = r(\theta)$;
- Objection-(2a): $r(\theta)$ is a representation of no / low information;
(2b) Multiple choices for reasonable reference densities: Laplace's (flat), Jeffreys (invariant), Maximum Entropy (under different constraints), Amari's (info. geometry), Bernardo's, etc.
– an opportunity for sensitivity / robustness analysis (2004).
- "Bug as a feature" losses: No artificial measures or ad hoc weights on $H$ , no nuisance parameter elimination tricks, etc.

- FBST is Not a Bayesian procedure (Jose Bernardo):

– FBST can be derived from a Loss / Utility function (2001)

• M.R.Madruga, L.G.Esteves, S.Wechsler (2001). On the Bayesianity of Pereira-Stern tests. *Test.* • M.R.Madruga, C.A.B.Pereira, J.M.Stern (2003). Bayesian Evidence Test for Precise Hypotheses. *J. Stat. Plan. Inference.*
• J.M. Stern (2004). Paraconsistent Sensitivity Analysis for Bayesian... *LNAI*

# FAQ1: e-value / FBST objections - Theoretical

- Objection: ev $(H \mid X)$ is Not a PROBABILITY (calculus)!
- Dennis Lindley letter to Carlos Pereira of 04/005/2006;
- John Skilling in MaxEnt reviews of 29-30/07/2017.

- Indeed, ev $(H \mid X)$ is a POSSIBILITY calculus / measure in $\mathcal{H}$, the hypotheses' space, derived from $p_n(\theta \mid X)$ , in $\Theta$ , derived in accordance to Likelihood principle, full Invariance, no *adhoc*-s;
- Probability $\leftrightarrow$ Possiblity transformations theory & analysis in *Log. J. IGPL* (2014): Focus on Surprise, Measure Probability!
- GFBST decisions are in strict accordance with *Modal Logic* inference principles i.e. intuitive and understandable rules for reasoning and clear (natural grammar) communication;
- FBST decisions agree with legal principles of *Onus Probandi* & *In Dubio Pro Reo*, see *F. Artif. Intell.* (2003): Burden of Proof...

- The right tool for the right job! ev $(H) \not\approx$ subjective / collective responsibility / punishment legal liability doctrine at case study 'the gatecrasher' in D. Lindley (1991), *J. Royal Stat.*, 154, 83-92; see Jonathan Cohen (1977) *The Probable and the Provable*.

- Edward Batschelet (1975). *Introduction to Mathematics for Life Scientists*.

-1/0
- Bruno de Finetti (1957). *Matematica Logico-Intuitiva*. Roma: Cremonese.
- David Murdoch (1967). *Analytic Geometry w.Introd. to Vectors & Matrices*

0
- Morris H. DeGroot, Mark J. Schervish (2012). *Probability and Statistics*.

- J. Kemeny, L. Snell, G. Thompson (1974). *Introd. to Finite Mathematics*.

-1
- M.Marcus (1969). *A Survey of Finite Mathematics*. ↪ Comp. Program.
- Lev V. Tarasov (1982). *Calculus: Basic Concepts for High Schools*. MIR.
- Fred Attneave (1959). *Applications of Information Theory to Psychology*.

- Yuri A. Rozanov (1977). *Probability Theory: A Concise Course*. Dover.

0/+1
- A.I. Khinchin (1957). *Mathematical Foundations of Information Theory*.
- A.J.Pettofrezzo (1966), *Vectors & Applictns.* / *Matrices & Transformations*.
- R.J. Goult (1978), *Applied Linear Algebra*; (1974), *Comput. Meth. in Lin. Alg*.
- Harley Flanders (1974), *A* $1^{st}$, $2^{nd}$ *Course in Calculus w. Analytic Geometry;*
- Nikolai S. Piskunov (1969). *Differential and Integral Calculus*. MIR.
- A.Mood, F.Graybill, D.Boes (1974). *Introduction to the Theory of Statistics.*
- Hoel, Port, Stone (1973). Intr. Probab. / Statistical Th. / Stoch. Processes.
- János Aczél (1966). *Lect. on Functional Equations and their Applications*.
- Germund Dahlquist, Ake Bjorck (1974). *Numerical Methods*. Prentice-Hall.

- A. Gelman, J. Carlin, H. Stern, D. Rubin (2013). *Bayesian Data Analysis*.

2+
- William Feller (1966). *An Introd. to Probability Theory and Its Applications*.
- Tom Mike Apostol (1967), *Calculus*, v. I & II; (1965), *Mathematical Analysis*.
- Lorenzo Sadun (2007). *Applied Linear Algebra: The Decoupling Principle*.