Site Percolation on a Disordered Triangulation of the Square Lattice

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Abstract

In this note we consider independent site percolation in a disordered triangulation of $\mathbb{R}^2$ given by adding one of the two possible diagonals to each face of the usual graph $\mathbb{Z}^2$. The natural conjecture is that $p_c = \frac{1}{2}$ for every such graph. Bollobás and Riordan proved this for almost every triangulation, in case each diagonal is chosen independently using a given coin. We give an alternative proof to the particular case of a fair coin. The general conjecture remains open.

1 Introduction

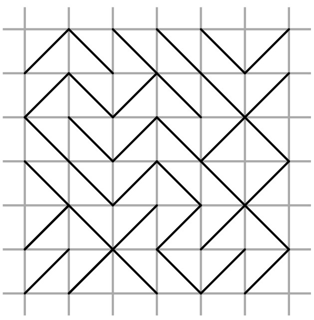
Let $\widetilde{\mathbb{Z}}^2$ denote the set of 1×1 squares in $\mathbb{R}^2$ having all its corners in $\mathbb{Z}^2$, and define the diagonal configuration space by

$$\Omega = \{ \boxbslash, \boxslash \}^{\widetilde{\mathbb{Z}}^2}.$$

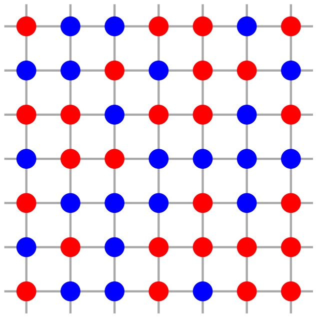
Let Σ denote the color configuration space

$$\Sigma = \{ {\color{blue}\bullet}, {\color{red}\bullet} \}^{\mathbb{Z}^2}.$$

Examples of a diagonal configuration ω and a color configuration σ are



and



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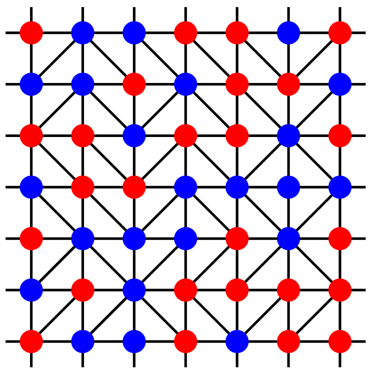
Let P_p denote the probability measure on Σ given by

$$P_p(\sigma_x = {\color{red}\bullet}) = {\color{red}\boldsymbol{p}} \qquad \text{and} \qquad P_p(\sigma_x = {\color{blue}\bullet}) = {\color{blue}\mathbf{1}-\boldsymbol{p}},$$

independently over $x \in \mathbb{Z}^2$, and let P_p^ω denote the law of the percolation process on the graph obtained by adding the diagonals in ω to the usual graph $\mathbb{Z}^2$. In general, the resulting graph does not have any symmetry, but still there is one critical parameter $p_c(\omega)$ at which the probability of having an infinite red cluster jumps from 0 to 1 (this is a tail event).

Since this graph is a triangulation, site percolation is self-dual, that is, the only way to prevent a given red connection is with a transversal blue connection and vice-versa. This is illustrated by


,

where existence of a left-right red crossing prevents a top-bottom blue crossing, and a left-right blue crossing prevents a top-bottom red crossing.

1.1 Statement

Because of self-duality, the obvious conjecture[1] is that $p_c(\omega) = \frac{1}{2}$. In this note we show that this is true if the diagonal configuration ω is obtained by tossing a fair coin for each square $z \in \widetilde{\mathbb{Z}}^2$.

Let Q denote the probability on Ω given by

$$Q\big(\omega_z = \boxbslash\big) = Q\big(\omega_z = \boxslash\big) = \tfrac{1}{2},$$

independently over different squares $z \in \widetilde{\mathbb{Z}}^2$. Our sample space will be $\Omega \times \Sigma$, so the process described above is governed by the "quenched measure"

$$P_p^\omega = \delta_\omega \times P_p.$$

[1]If p_c were smaller, there would be both an infinite blue and an infinite red cluster for any $p \in (p_c, 1-p_c)$, and if p_c were larger, there would be both infinitely many red and blue circuits surrounding the origin for any $p \in (1-p_c, p_c)$.

The "annealed measure" is given by

$$\mathbb{P}_p = Q \times P_p = \int_\Omega P_p^\omega \, \mathrm{d}Q(\omega).$$

Theorem 1. *For the annealed process,*

$$\mathbb{P}_p(\textit{there is an infinite red cluster}) = \begin{cases} 0, & p \leqslant \frac{1}{2}, \\ 1, & p > \frac{1}{2}. \end{cases}$$

In particular, by Fubini-Tonelli, $p_c(\omega) = \frac{1}{2}$ *for almost every* ω*.*

1.2 Related work and acknowledgements

Bollobás and Riordan [2] proved a more general result, where the measure Q can be taken as i.i.d. with any parameter. The author was unaware of that work until recently.

The alternative argument presented here is elementary and self-contained. It consists in adapting Smirnov's proof of RSW estimates [6, 7] and Kesten's proof of logarithmic expected number of pivotal sites [4]. The argument given in [2] is more general but slightly more involved, as it uses uniqueness of the infinite cluster and requires adapting Menshikov's sausage argument [5].

An important step in the proof is to establish the analogous of Harris-FKG inequality for the annealed measure. Notice that not all events which we would normally call "increasing" are positively correlated, because two such events may have conflicting requirements for the diagonals. To overcome this, we define *robust increasing* events, and prove positive correlations for this type of event. The same idea was already present in [2], and we give a precise description here for the reader's convenience.

Recent work on Bernoulli percolation in $\mathbb{Z}^d$ and Voronoi percolation in $\mathbb{R}^2$ provided new insights for the study of sharp phase transitions [1, 3, 8]. It would be interesting to apply those ideas to disordered triangulations of the square lattice, and obtain alternative proofs or extend existing results to a more general setting. In particular, it seems that the arguments introduced in [1] to prove a quenched version of RSW estimates for Voronoi percolation would also work for disordered triangulations of the square lattice.

I would like to thank Wendelin Werner for suggesting this problem to me back in 2009, and for inspiring discussions. I also thank him for pointing out the possible connections with [3] and [8]. I thank Vincent Tassion for pointing out reference [2] and the RSW estimates from [1].

2 Robust increasing events

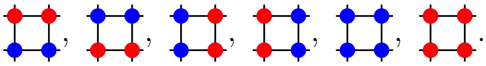
An *observable* is a measurable function $f : \Omega \times \Sigma \to \mathbb{R}$.

Definition. We say that an observable f is *increasing in* σ if, for each pair (ω, σ), switching the color of any site x from $\color{blue}\bullet$ to $\color{red}\bullet$ increases (i.e., does not decrease) the value of f.

In order to discuss monotonicity with respect to the diagonal configuration ω, we take into account the color configuration σ to see whether it is $\boxbslash$ or $\boxslash$ who favors the $\color{red}\bullet$'s more than the $\color{blue}\bullet$'s, or the other way around.

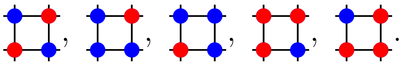
Given a color configuration σ, each square $z \in \widetilde{\mathbb{Z}}^2$ will be classified as having one of three types, depending on the colors of its four corners.

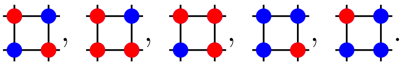
The first type consists of configurations whose symmetries make it impossible to decide whether $\color{red}\bullet$'s and $\color{blue}\bullet$'s would prefer $\boxbslash$ or $\boxslash$.

– type N: 

Definition. We say that f is *robust* if flipping the diagonal at any square z of type N does not change the value of f.

Squares that are not of type N will have type A or B depending on whether it is the $\color{red}\bullet$'s or $\color{blue}\bullet$'s who prefer $\boxslash$ over $\boxbslash$.

– type A: 

– type B: 

Definition. We say that a robust observable f is *increasing in* ω if, for each pair (ω, σ), flipping the diagonal of any square z from $\boxbslash$ to $\boxslash$ will increase f for z of type A and decrease f for z of type B. An event $\mathcal{A}$ is called robust, increasing in σ and ω if its indicator function is so.

We mention that some events that would normally be called "increasing" are not robust. For example, in a 3×3 square (what we call 3×3 contains 16 sites), existence of a red path of length 3 connecting the top-left and bottom-right corners is not robust. Moreover, this event is not positively-correlated with existence of a red path of length 3 connecting the top-right and bottom-left corners: they are in fact mutually exclusive. The above events are not robust because they have requirements for diagonals even when the containing squares are of type N. In the same direction, events requiring existence of disjoint paths are not robust in general.

On the other hand, and that is enough for our needs, for any sets $A, B \subseteq \mathbb{Z}^2$ and any domain $\mathcal{D}$ consisting of a collection of closed squares of $\widetilde{\mathbb{Z}}^2$, the event "A is connected to B by a red path in $\mathcal{D}$" is both robust and increasing.

Lemma 1 (Harris-FKG)**.** *Let f and g be robust non-negative observables, increasing in σ and ω. Then*

$$\mathbb{P}_p(fg) \geqslant \mathbb{P}_p(f)\mathbb{P}_p(g).$$

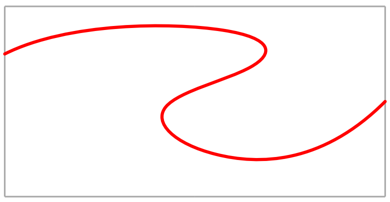
Proof. Let $\sigma \in \Sigma$ be fixed. For an observable h, consider the projection $h_\sigma : \Omega \to \mathbb{R}$ given by $h_\sigma = h(\cdot, \sigma)$. Observe that, if h is robust and increasing in ω, then h_σ depends only on ω_z for z outside the set $N_\sigma \subseteq \widetilde{\mathbb{Z}}^2$ of squares of type N. Moreover, there is a natural partial order on $\{\boxbslash, \boxslash\}^{N_\sigma^c}$ under which h_σ is an increasing function ($\boxbslash \leq \boxslash$ for $z \in N_\sigma^c$ of type A and $\boxbslash \geq \boxslash$ for $z \in N_\sigma^c$ of type B). Since Q induces a product measure on $\{\boxbslash, \boxslash\}^{N_\sigma^c}$, projections of this type satisfy the Harris-FKG inequality with respect to Q. Therefore, if f and g are non-negative, robust and increasing in σ and ω,

$$\mathbb{P}_p(fg) = P_p[Q(f_\sigma g_\sigma)] \geqslant P_p[Q(f_\sigma)Q(g_\sigma)] \geqslant P_p[Q(f_\sigma)]P_p[Q(g_\sigma)] = \mathbb{P}_p(f)\mathbb{P}_p(g).$$

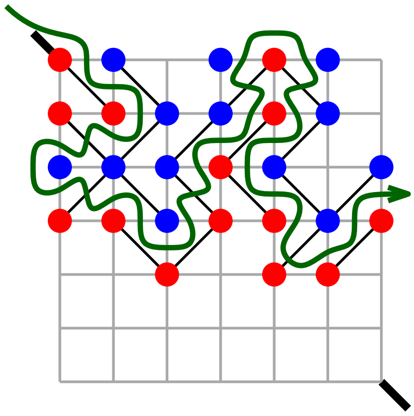
We have used Fubini-Tonelli theorem for the equalities. The first inequality follows from the above observation and the second inequality follows from the standard Harris-FKG inequality, since f and g are increasing in σ. □

3 Proof of sharp percolation threshold

Lemma 2 (Russo-Seymour-Welsh)**.** *In any $2n \times n$ rectangle,*

$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geqslant \frac{1}{16}.$

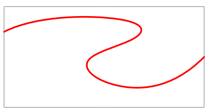
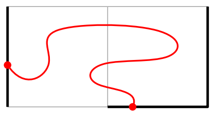
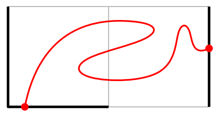
In the proof we consider an exploration that progressively reveals the color of some sites and the position of some diagonals. Below we show an exploration starts from the top-left corner and targets the bottom-right corner of a rectangle. When the exploration enters a triangle by crossing one of its sides, it looks at the color of the opposite corner in order to decide on where to exit the triangle. When it enters a square, it first reveals the position of the diagonal on that square.



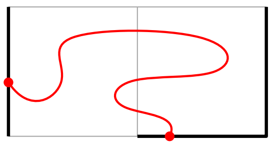
(1)

In this procedure, the exploration will leave the rectangle through the right side before the bottom side if there is a left-right red connection, and the bottom side before the right otherwise.

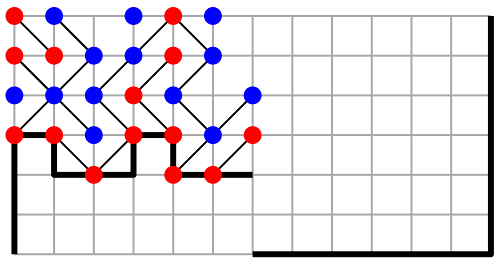
Proof of Lemma 2. We use $\mathbb{P}$ for $\mathbb{P}_{\frac{1}{2}}$. By Harris-FKG we have

$\mathbb{P}\Big($  $\Big) \geqslant \mathbb{P}\Big($  $\Big) \times \mathbb{P}\Big($  $\Big)$,

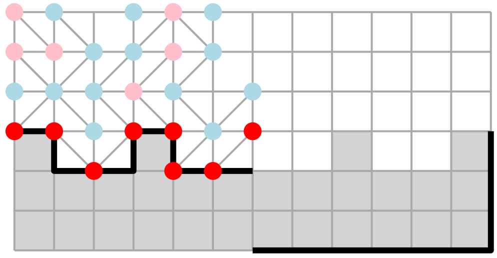
whence by symmetry it suffices to show that

$\mathbb{P}\Big($  $\Big) \geqslant \frac{1}{4}$. (2)

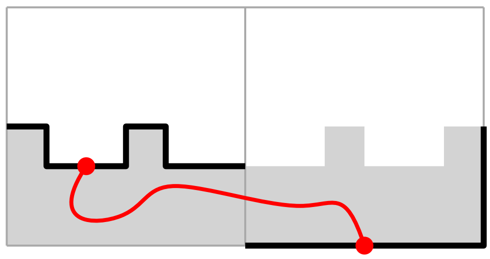
We first try to "bend" the left-side boldface region by starting an exploration path in the left-side square as shown in (1). With probability $\frac{1}{2}$ we succeed bending the boldface region until the middle of the rectangle, revealing some diagonals and some blue and red sites like this:

 .

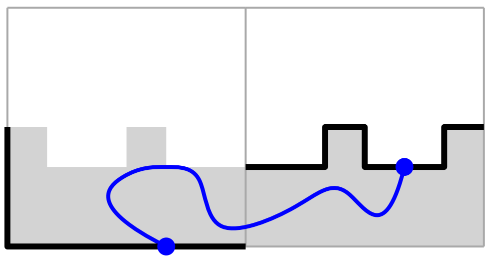
Given a partial configuration such as above, the event in (2) is equivalent to a red connection between the two boldface regions. So we need to show that the conditional probability of such red connection is at least $\frac{1}{2}$. But such connection is certainly implied by red path connecting two smaller boldface regions contained in a smaller grayed zone given by

 .

Now notice that none of the sites and diagonals revealed so far can interfere with this event, except for some red sites lying on the boldface region. The fact that these sites are red can only help, and the conditional probability of the latter event given that they are red is bounded from below by the probability of the crossing



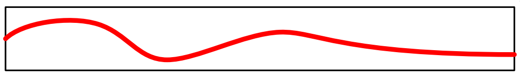
without any conditioning. Finally, the complementary of the latter event is

 ,

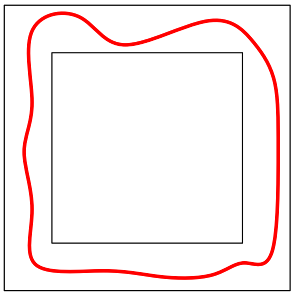
which by symmetry has the same probability, concluding the proof. □

As usual, Lemma 2 has the following immediate corollaries.

Corollary 1. *There is $\delta > 0$ such that, in any $8n \times n$ rectangle,*

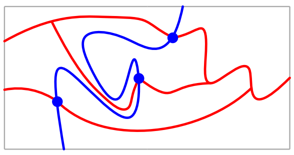
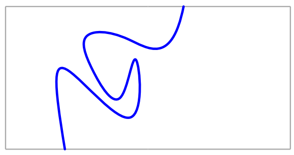
$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geqslant \delta.$

Corollary 2. *There is $\delta > 0$ such that, in any pair of co-centered squares of size $4n \times 4n$ and $6n \times 6n$,*

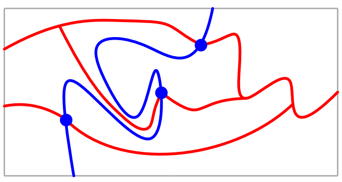
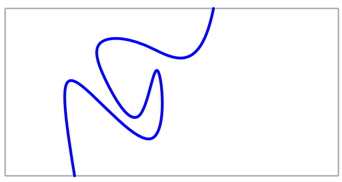
$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geqslant \delta,$

The last piece in the proof is the following.

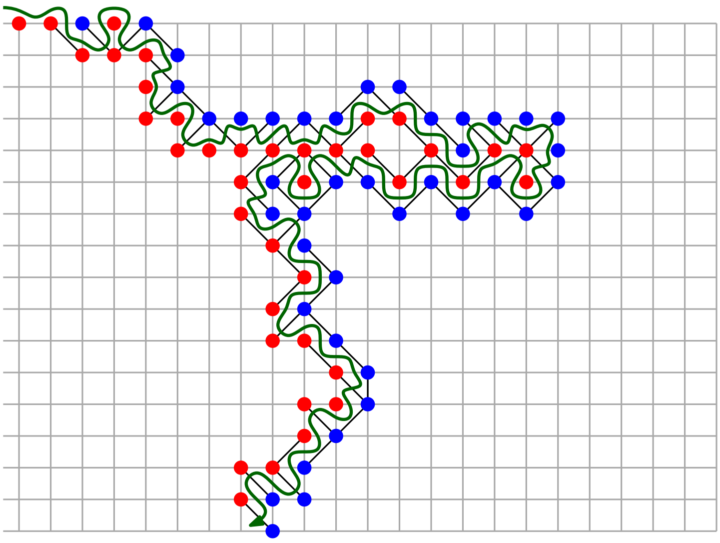
Lemma 3 (Kesten)**.** *There is $\beta > 0$ such that, for any $p \geqslant \frac{1}{2}$, in any $2n \times n$ rectangle, the expected number of pivotal sites satisfies*

$\mathbb{E}_p\Big($  $\Big) \geqslant \beta \log_2 n \times \mathbb{P}_p\Big($  $\Big).$

Proof. It suffices to show that

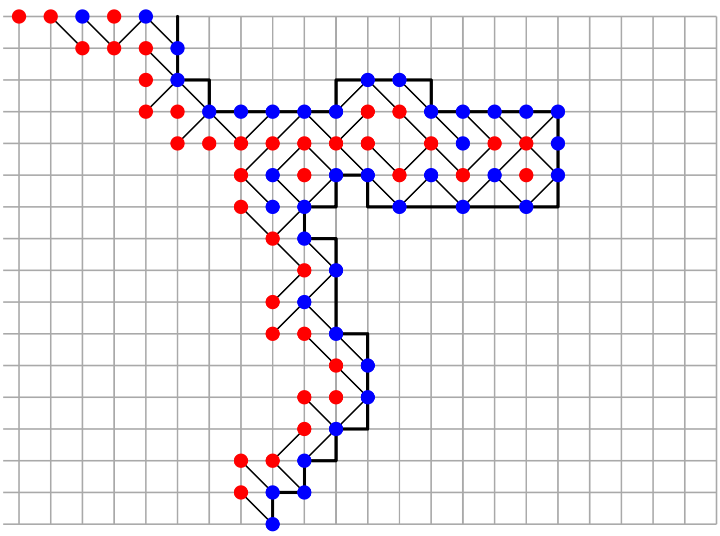
$\mathbb{E}_p\Big($  $\Big|$  $\Big) \geqslant \beta \log_2 n.$

We will determine occurrence of the top-bottom blue crossing using an exploration path that starts at the top-left corner and ends at either the bottom or the right side, as below.



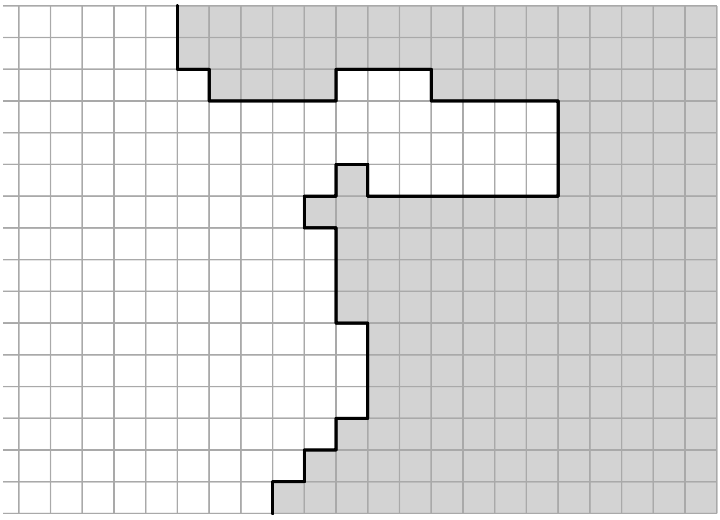
Existence of a top-bottom blue crossing is equivalent to the exploration finding the bottom side before the right side. We want to show that, if it such crossing occurs, then the conditional expectation of the number of pivotal sites given the colors and diagonals revealed in this exploration is greater than $\beta \log_2 n$ for some constant β.

On the above event, there is a self-avoiding blue path that joins the top and bottom sides of the rectangle. What we do now is a little overkilling, but it avoids the hassle of considering all corner cases related to the diagonals. Let us first inflate this self-avoiding blue path to make it squared where it would otherwise use a diagonal, as below.



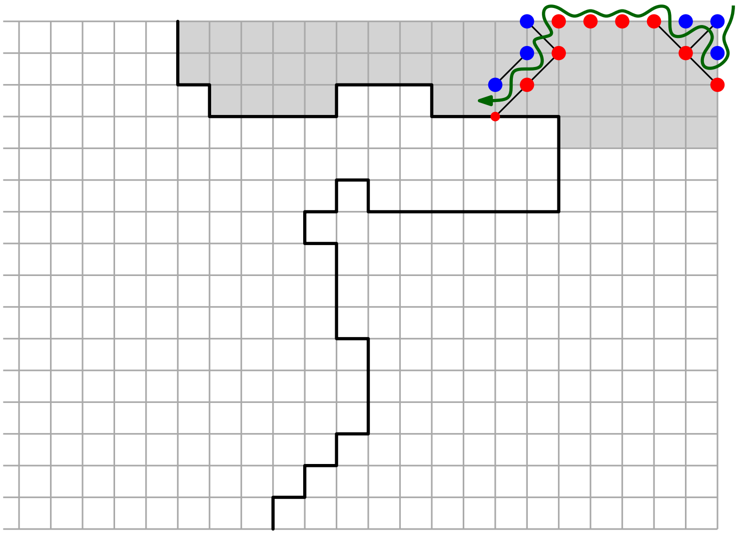
Notice that there are two types of sites in this squared path. The first type consists of blue sites which are adjacent to a red site which is in turn connected to the left side of the rectangle by a red path. The second type consists of sites whose color has not yet been revealed, but which are adjacent to a site of the first type.

Now, in order to find pivotal sites, we consider the domain consisting of the squares that can be reached from the right side of the big rectangle without crossing the squared path, colored in light-gray below.



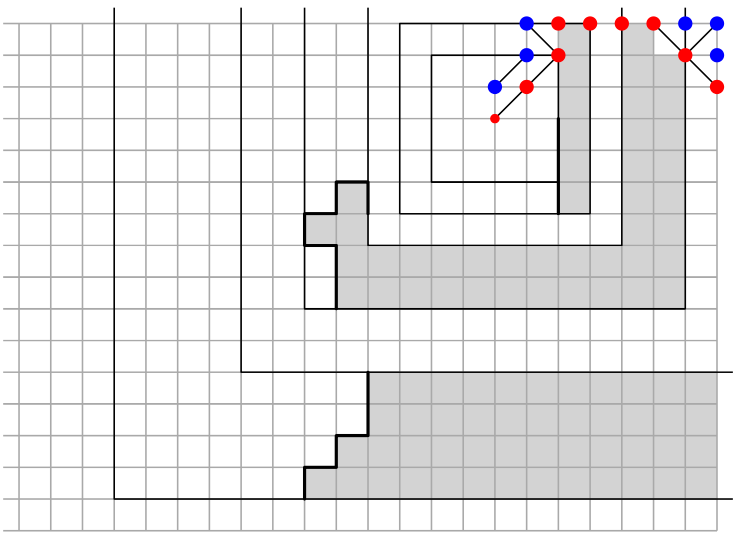
We will "re-sample" the entire process on this domain. Of course we cannot do that, so the red sites that we find on the squared path that turn out to have been previously sampled as blue will remain blue in the end.

We first look for a connection from the right side of the big rectangle to the squared-path that stays in a strip of width $\frac{n}{4}$ using an exploration path as below.



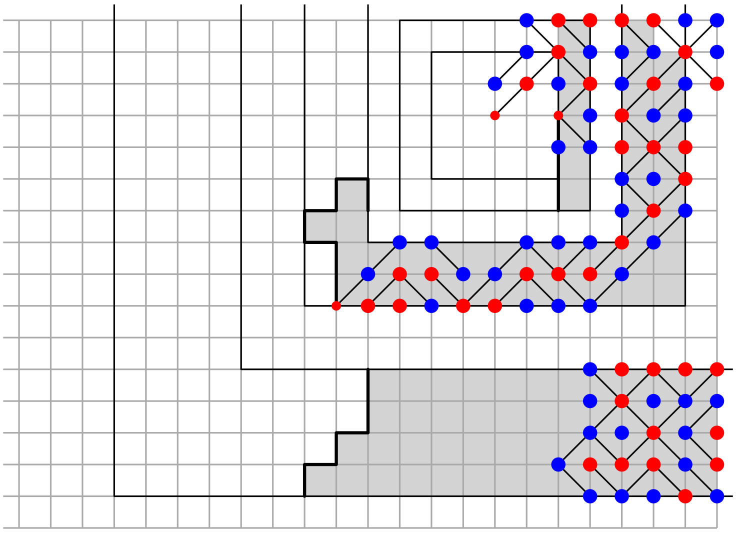
The last red site was drawn smaller to remind us that it may be a site which we already revealed to be blue. By Corollary 1, the probability of finding such a connection is greater than δ.

In case such path is found, we now draw disjoint $4 \cdot 2^k$-sided and $6 \cdot 2^k$-sided squares centered at the point just found, and consider the intersection of the corresponding annuli with the light-gray region in the second previous picture. There are at least $\delta \log_2 n$ such regions that do not go lower than the bottom side of the big rectangle, minus the squares explored in the previous step, as shown below.

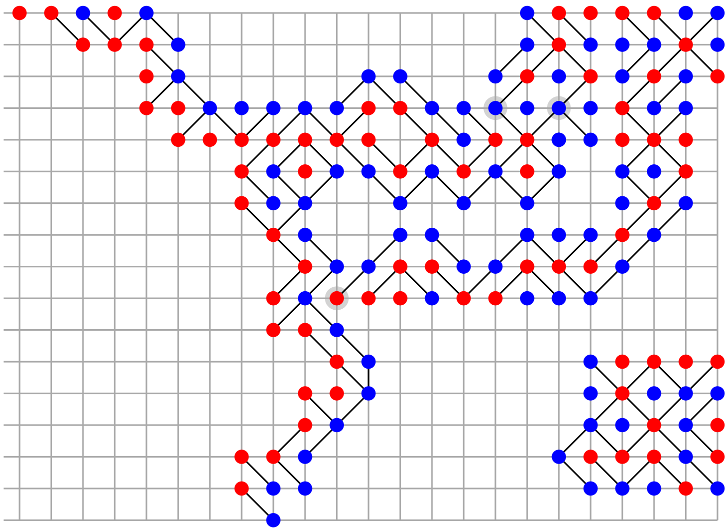


Conditioned on the above picture, by Corollary 2 each of these "tunnels" will contain a red connection with probability at least δ. The conditional expectation for the number of such tunnels that actually contain such connection, given that the previous connection has been found, is thus greater than $\delta^2 \log_2 n$. Therefore, the conditional expectation given existence of a top-bottom blue crossing is greater than $\delta^3 \log_2 n$.

In the example below, two such tunnels ended up providing one red site (drawn smaller), and one of them did not.

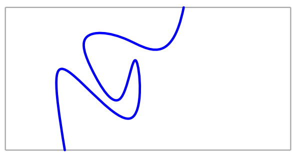


We combine the configuration discovered at this stage with the one previously removed. At this point, a small red site will become a true red site if it had not been revealed in the first exploration, and will be reverted to blue in case it had. The result is highlighted by a light-gray disk below.



To conclude the proof, notice that each of these highlighted sites is either a pivotal site in case it is blue, or is preceded by a pivotal site in the squared curve in case it is red. $\square$

Proof of Theorem 1. Absence of percolation at $p = \frac{1}{2}$ follows from Corollary 2 as usual. Define the event

$$\mathcal{A}_n =$$ 

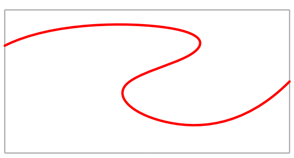
in a $2n \times n$ rectangle. Using Russo's formula and Lemma 3 we get

$$\tfrac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(\mathcal{A}_n) = -\mathbb{E}_p\big[\,\text{pivotal sites}\,\big] \leqslant -\beta \log_2 n\, \mathbb{P}_p(\mathcal{A}_n),$$

which gives

$$\log \mathbb{P}_{\frac{1}{2}+\varepsilon}(\mathcal{A}_n) \leqslant -\varepsilon\beta \log_2 n,$$

and thus, on an $2^k \times 2^{k+1}$ rectangle,

$$\mathbb{P}_{\frac{1}{2}+\varepsilon}\Bigg($$  $$\Bigg) \geqslant 1 - e^{-\varepsilon\beta k}.$$

To conclude the proof, we arrange $2^k \times 2^{k+1}$ rectangles as

and deduce the second part of the theorem using Harris-FKG inequality. □

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