Cayley's Theorem

Notes for Chapters 1 and 19

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Abstract

A square matrix $\mathbf{A} = (a_{ij})$ of order n is *skew-symmetric* if $a_{ij} = -a_{ji}$, for each $1 \leq i \leq j \leq n$. A well-known theorem of Cayley states that the determinant of a skew-symmetric matrix $\mathbf{A}$ is a perfect square. It is easy to see that when n is odd, the determinant of $\mathbf{A}$ is in fact zero (see Exercise 2.1). When n is even, it turns out that $\det(\mathbf{A})$ is the square of a polynomial in the entries of $\mathbf{A}$ known as the *Pfaffian* of $\mathbf{A}$. Each term in the Pfaffian of $\mathbf{A}$, denoted by $\mathrm{Pf}(\mathbf{A})$, corresponds to a perfect matching in a graph $G(\mathbf{A})$ that is associated with $\mathbf{A}$. This correspondence leads to the notion of a Pfaffian orientation of a graph. Pfaffian orientations are the subject of Part III of our book.

1 Digraph Representations of Square Matrices

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n, and let $\{1, 2, \ldots, n\}$ be the index set of the rows and columns of $\mathbf{A}$. The standard way of representing $\mathbf{A}$ is by an $n \times n$ array. But $\mathbf{A}$ can be visualized by means of a weighted digraph $D^*(\mathbf{A})$ on $\{1, 2, \ldots, n\}$ in which, there is a loop at each vertex i with attached weight a_{ii}, and between any two distinct vertices i and j, there are arcs (i, j) and (j, i) with attached weights a_{ij} and a_{ji}, respectively. (When there is no scope for confusion, we shall simply write D^* for $D^*(\mathbf{A})$.) Observe that, in this digraph D^*, for any vertex i, the weights of the arcs with i as their tail correspond to the entries in the i^{th} row of $\mathbf{A}$, and the weights of the arcs with i as their head correspond to the entries in its i^{th} column.

A *transversal* of $\mathbf{A}$ is a selection of n entries of $\mathbf{A}$ no two of which belong to the same row or the same column. Consider any transversal T. Since T has n entries, it corresponds in D^* to a set of n arcs such that, at each vertex i, there is precisely one incoming arc with i as its head, and precisely one outgoing arc with i as its tail. This leads us to the following pivotal observation:

Proposition 1.1 *The subdigraph of D^* induced by the set of arcs in a transversal T of $\mathbf{A}$ is a union of vertex disjoint directed cycles which covers all the vertices of D^*.* □

We refer to this subdigraph of D^* corresponding to a transversal T of $\mathbf{A}$ as the *support* of T. The above proposition says that all the *components of the support* of a transversal are directed cycles.

We now define the notions of weights and signs of transversals of $\mathbf{A}$, and use them to define the determinant $\det(\mathbf{A})$ of $\mathbf{A}$.

Let C be a directed cycle in the digraph D^*. Then the *weight* $\mathrm{wt}(C)$ of C in D^* is the product of the weights of the the arcs in C. The *sign* of C, denoted by $\mathrm{sign}(C)$, is plus if the length of C is odd, and is minus if that length is even. Now suppose that T is a transversal of $\mathbf{A}$. Then, the *weight* $\mathrm{wt}(T)$ of T is the product of the weights of the directed cycles whose union is the support of T. Similarly, the sign of T, denoted by $\mathrm{sign}(T)$, is the product of the signs of the directed cycles whose union is the support of T.

DEFINITION OF $\det(\mathbf{A})$ IN TERMS OF D^*

The *determinant* $\det(\mathbf{A})$ of $\mathbf{A}$ is $\sum_T \mathrm{sign}(T)\mathrm{wt}(T)$, where the sum is taken over the set of all transversals T of $\mathbf{A}$. (Exercise 1.1 provides a simple illustration of how to compute the determinant of a matrix using this definition.)

Notes: The definition of the determinant of a square matrix is usually phrased in terms of permutations. To see that the above given definition is equivalent to the 'standard' definition, observe that $T := \{a_{1\tau(1)}, a_{2\tau(2)}, \ldots, a_{n\tau(n)}\}$ is a transversal of $\mathbf{A}$ if and only if τ is a permutation of $\{1, 2, \ldots, n\}$. Thus, there is a one-to-one correspondence between the set of all transversals of $\mathbf{A}$ and the set of all the $n!$ permutations of $\{1, 2, \ldots, n\}$. Proposition 1.1 implies that any permutation of $\{1, 2, \ldots, n\}$ may be decomposed into cyclic permutations. The sign of a cyclic permutation is plus or minus depending whether or not it is odd or even. Thus the above definition of the determinant of $\mathbf{A}$ is the same as the definition using the language of permutations. In the first three sections, we shall restrict ourselves to the language of graph theory. But in the last two sections, we will need to resort to the terminology of the theory of permutations which is commonly used (including our own book *Perfect Matchings*). Most undergraduate texts on algebra include the basic material concerning permutations we need. But, for the convenience of our readers, we give a brief review of all the facts concerning permutations that we use in the appendix at the end.

Exercise

1.1 *Consider the skew-symmetric matrix* $\mathbf{A}$ *of order four in which the six upper diagonal entries* $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}$ *and* a_{34} *are positive, the six lower diagonal entries* $a_{21} = -a_{12}, a_{31} = -a_{13}, a_{41} = -a_{14}, a_{32} = -a_{23}, a_{42} = -a_{24}$, *and* $a_{43} = -a_{34}$ *are negative, and the four diagonal entries* a_{11}, a_{22}, a_{33} *and* a_{44} *are zero.*

(i) *Draw the digraph* $D^*(\mathbf{A})$. *(There is no need to draw the loops at the four vertices because any transversal which includes a diagonal entry has zero weight and thus makes no contribution to* $\det(\mathbf{A})$.*)*

(ii) *Find all the transversals of* $\mathbf{A}$ *of nonzero weight and their signs and weights. Hint: There are three transversals of nonzero weight whose supports are unions of two directed cycles of length two in* $D^*(\mathbf{A})$. *One such transversal is* $\{a_{12}, a_{21}, a_{34}, a_{43}\}$, *its sign is plus, and its weight is*

$$a_{12}.a_{21}.a_{34}.a_{43} = a_{12}.(-a_{12}).a_{34}.(-a_{34}) = a_{12}^2.a_{34}^2.$$

There are six transversals whose supports are single directed cycles of length four, which come in pairs that pertain to directed cycles in $D^*(\mathbf{A})$ *which are* *converses of each other. For example, the cycles* $(1,2,3,4,1)$ *and* $(1,4,3,2,1)$ *are two such directed cycles in* D^**; the signs of the transversals which correspond to these two cycles are minus, and their weights are* $a_{12}.a_{23}.a_{34}.a_{41}$ *and* $a_{14}.a_{43}.a_{32}.a_{21}$*, respectively. Using the skew-symmetry property, show that the weights of both these directed cycles are equal to* $-a_{12}.a_{23}.a_{34}.a_{14}$*.*

(iii) *Now find the* $\det(\mathbf{A})$ *and verify that it is the square of*

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

2 Representation of Skew-Symmetric Matrices by Simple Digraphs

Suppose that $\mathbf{A}$ is a skew-symmetric matrix. Then, by definition, $a_{ij} = -a_{ji}$ for $1 \leq i \leq j \leq n$. This implies that all the diagonal entries of $\mathbf{A}$ are zero. The weight of any transversal of $\mathbf{A}$ which includes a diagonal entry or, more generally, any transversal of $\mathbf{A}$ which includes an entry that is zero, makes no contribution to $\det(\mathbf{A})$. For this reason, in $D^*(\mathbf{A})$ we retain only those arcs which correspond to nonzero entries of $\mathbf{A}$. Half the arcs of $D^*(\mathbf{A})$ corresponds to positive entries of $\mathbf{A}$ and, in view of the fact that $\mathbf{A}$ is skew-symmetric, the remaining arcs of $\mathbf{A}$ correspond to negative entries of $\mathbf{A}$. We denote the spanning weighted simple subdigraph of D^* induced by the set of arcs corresponding to the positive entries of $\mathbf{A}$ by $D := D(\mathbf{A})$, and the underlying weighted simple undirected graph of $D(\mathbf{A})$ by $G := G(\mathbf{A})$. **The weights of all edges in G are positive.**

Example 2.1 *The weighted digraph* D *depicted in Figure 1 represents a skew-symmetric matrix* $\mathbf{A}$ *of order ten with 15 positive entries corresponding to the arcs of* D*. In this case the underlying undirected graph* G *is the 5-prism. (To obtain* D^* *from* D*, for each arc* (i,j) *of* D *we simply have to add the reverse arc* (j,i) *and assign it the weight* $-a_{ij}$*.)*

2.1 Definitions of transversals of A and their signs and weights in terms of the simple digraph $D(\mathbf{A})$

In order for us to transition from D^* to D, we must first identify what happens to directed cycles of D^* when all the arcs with negative weights are deleted. If $(v_1, v_2, \ldots, v_{2k-1}, v_{2k}, v_1)$ is a directed cycle in D^* of length three or more, then $(v_1, v_{2k}, v_{2k-1}, \ldots, v_2, v_1))$ is also a directed cycle in D^*. These two directed cycles in D^* correspond to the same cycle in D traversed in two opposite senses. For convenience, we refer to the two possible senses of traversal of a cycle γ in D as 'clockwise' and 'anti-clockwise' and denote them by γ^c and γ^a, respectively (even when these designations have no geometric context).

The *weight* of a cycle in D with a prescribed sense of traversal is the product of the weights of those of its arcs whose orientations are also in the same sense and the negatives of the weights of those whose orientations are in the opposite sense. For example, let γ be the 5-cycle bounding the outer face in the drawing of the pentagonal prism shown in Figure 1. When γ is traversed in the clockwise sense, it encounters the heads of the four arcs $(2,1), (3,2), (4,3)$, and $(5,4)$ first

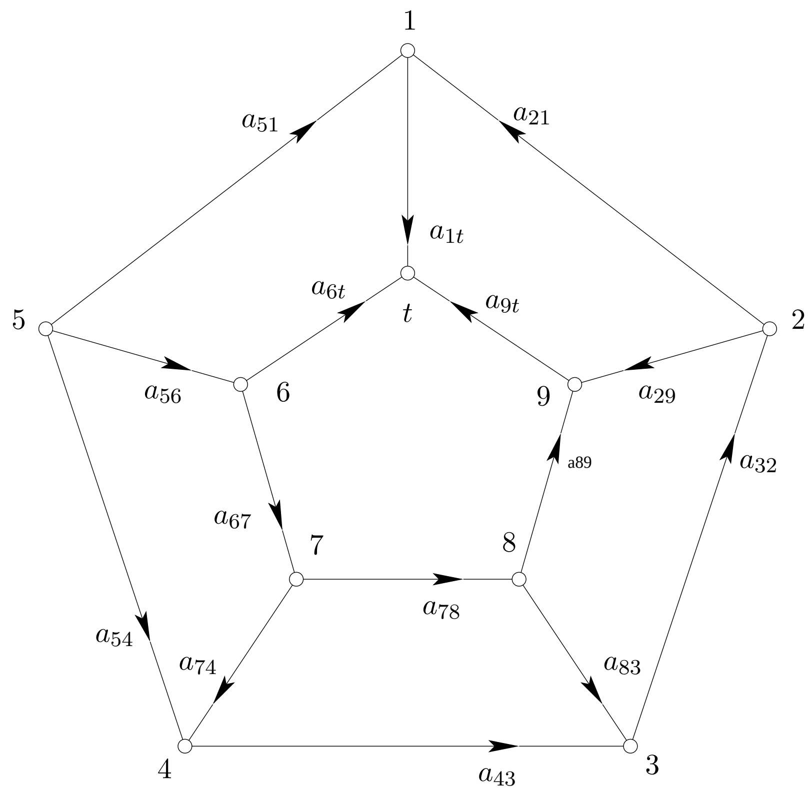


Figure 1: The weighted digraph D associated with a skew-symmetric matrix.

and then their tails later, and the tail of $(5,1)$ first then its head. Therefore, the weight of γ^c is $(-a_{21}).(-a_{32}).(-a_{43}).(-a_{54}).a_{51}$. On the other hand, the only arc of $\gamma^a := (1,5,4,3,2,1)$ whose orientation is not in the anti-clockwise sense is $(5,1)$. Therefore, the weight of γ^a is $(-a_{51}).a_{54}.a_{43}.a_{32}.a_{21}$.

Since D is simple, it has no cycles of length two. But, given any arc (i,j) of D, there is a unique directed cycle of length two in D^*, namely (i,j,i) and its weight is $a_{ij}.a_{ji} = a_{ij}.(-a_{ij}) = -a_{ij}^2$, and its sign is minus because it corresponds to an even cycle in D^*. For each arc (i,j) in D, we regard (i,j,i) as a cycle (albeit a flattened cycle) and we represent it simply by (i,j).

If $\mathbf{A}$ is a skew-symmetric matrix and T is a transversal of $\mathbf{A}$, we observed in Section 1 that the support of T in D^* is a vertex-disjoint union of directed cycles which covers all its vertices. The weight of T is the product of the weights of directed cycles in its support, and the sign of T is plus if the number of even directed cycles in its support is even, and is minus if that number is odd. Weights and signs of T with respect to D may be defined similarly noting the exceptional case in which a subdigraph of D with just one arc corresponds to a directed cycle of length two in D^*, and that directed cycles of length three or more corespond in D to cycles with specified senses of traversal.

Example 2.2

Let $T := \{a_{1t}, a_{t6}, a_{65}, a_{51}, a_{29}, a_{98}, a_{83}, a_{32}, a_{74}, a_{47}\}$ be a transversal of the skew-symmetric matrix $\mathbf{A}$ represented by the graph D in Figure 1. The support of T in D^* is the union of the the three directed cycles $\gamma_1 := (1, t, 6, 5, 1)$, $\gamma_2 := (2, 9, 8, 3, 2)$, and $\gamma_3 := (7, 4, 7)$. The support of T in D is the union of the cycles $\gamma_1^c := (1, t, 6, 5, 1)$, $\gamma_2^a := (2, 9, 8, 3, 2)$ and $\gamma_3 := (7, 4, 7)$.

Between any two adjacent vertices in D^* there are two arcs; one whose weight is positive and one whose weight is negative. In D there are no arcs whose weights are negative, but one may traverse an arc from its head to its tail by 'paying the price'. In this world, the weight function is multiplicative and two wrongs make a right! Thus

$$\begin{aligned}
\mathrm{wt}(\gamma_1^c) &= a_{1t}a_{t6}a_{65}a_{51} = a_{1t}(-a_{6t})(-a_{56})a_{51} = a_{1t}a_{6t}a_{56}a_{51} \\
\mathrm{wt}(\gamma_2^a) &= a_{29}a_{98}a_{83}a_{32} = a_{29}(-a_{89})a_{83}a_{32} = -a_{29}a_{89}a_{83}a_{32} \\
\mathrm{wt}(\gamma_3) &= a_{74}a_{47} = a_{74}(-a_{74}) = -a_{74}^2; \text{ and} \\
\mathrm{wt}(T) &= \mathrm{wt}(\gamma_1^c)\mathrm{wt}(\gamma_2^a)\mathrm{wt}(\gamma_3)
\end{aligned}$$

Finally, note that the sign of T is minus because its support is a union of three (an odd number) of even cycles!

2.2 Forward arcs and reverse arcs

Let γ be a cycle in D, and let γ^c and γ^a be obtained by assigning opposite senses of traversal (clockwise and anti-clockwise) to γ. An arc (i, j) of γ is a *forward arc* in γ^c if, in the clockwise order, its tail precedes its head, and is a *reverse arc*, otherwise. Forward and reverse arcs in γ^a are similarly defined. Clearly, a forward arc of γ^c is a reverse arc of γ^a and vice versa.

Thus the weight of γ^c is the product of the weights of forward arcs and the negatives of the weights of reverse arcs in γ^c. Similar statement applies to γ^a. This observations lead us to the following useful result:

LEMMA 2.3 *Let γ be a cycle of length three or more in D, let C denote the cycle in G obtained by disregarding the orientations of the arcs in D, and let $\mathrm{wt}(C)$ denote the product of the weights of the edges of G of C (which is positive because weights of all edges in G are positive). Let r and f denote the the numbers, respectively, of reverse and forward arcs in γ^c. Then,*

$$\mathrm{wt}(\gamma^c) = (-1)^r \mathrm{wt}(C).$$

On the other hand, since those arcs of γ which are forward arcs in γ^c are reverse arcs in γ^a,

$$\mathrm{wt}(\gamma^a) = (-1)^f \mathrm{wt}(C). \qquad \square$$

(The interplay between cycles in the simple directed graph D and the cycles in the undirected graph G is an important aspect of the proof of Cayley's Theorem.)

If γ^c is a cycle of odd length, then r and f clearly have different parities because the length of γ^c is $r + f$. On the other hand, if the γ^c is of even length, then r and f have the same parity. Furthermore, forward arcs in γ^c are reverse arcs in γ^a, and vice versa. Thus, the above lemma implies the following useful property regarding the weights of cycles in D.

ODD AND EVEN CYCLES

COROLLARY 2.4 *Let γ be a cycle in G and let γ^c and γ^a denote cycles in D corresponding to the two senses of traversal of γ.*

(i) *if γ is an odd cycle, then $\mathrm{wt}(\gamma^a) = -\mathrm{wt}(\gamma^c)$; and*

(ii) *if γ is an even cycle, then $\mathrm{wt}(\gamma^a) = \mathrm{wt}(\gamma^c)$. (The fact that the weights of γ^c and γ^a are the same when γ is an even cycle might lead one to think that, in the expansion of $\det(\mathbf{A})$ one needs to consider only one of them. This is not true. Consider, for example, the cycle $\gamma := (1, 2, 3, 4, 1)$ in the digraph D representing the skew-symmetric matrix $\mathbf{A}$ of order four described in Exercise 1.1. Although γ^c and γ^a have the same the signs and same weights, they correspond to distinct transversals and both appear in the expansion of the determinant of the matrix $\mathbf{A}$.)* □

By Proposition 1.1, the support of a transversal cannot contain the same cycle with two different senses of traversal. The above result implies the following:

PROPOSITION 2.5 *Suppose that T and T' are two different transversals of a skew-symmetric matrix $\mathbf{A}$, and, in digraph D representing $\mathbf{A}$, let γ^c and γ^a be the same odd cycle with opposite senses of traversal such that (i) the support of T has γ^c as a component, (ii) the support of T' has γ^a as a component, and otherwise, (iii) every component of the support of T is also a component of the support of T', and vice versa. Then $\mathrm{wt}(T) = -\mathrm{wt}(T')$ (and $\mathrm{sign}(T) = \mathrm{sign}(T')$). (*So, in the expansion of the determinant of $\mathbf{A}$, the terms corresponding to T and T' cancel out.*)* □

For example, consider the skew-symmetric matrix represented by the digraph D shown in Figure 1, let γ_1 and γ_2 denote, respectively, the outer and inner facial pentagons, and let T and T' be the transversals whose support are, respectively, $\gamma_1^c \cup \gamma_2^c$ and $\gamma_1^c \cup \gamma_2^a$. Then $\mathrm{wt}(T) = -\mathrm{wt}(T')$ and signs of both T and T' are plus.

As we noted in the abstract the determinant of skew-matrix $\mathbf{A}$ of order n is zero if n is odd. This result can be deduced easily using Proposition 2.5 by observing that, as n is odd, the support of <u>any</u> transversal of $\mathbf{A}$ must contain an odd cycle. We leave the details as Exercise 2.1.

Henceforth, we restrict ourselves to the consideration of skew-symmetric matrices of even order. Proposition 2.5 leads us to the following **important conclusion**:

SKEW-SYMMETRIC MATRICES OF EVEN ORDER

THEOREM 2.6 The expansion of the determinant of any skew-symmetric matrix $\mathbf{A}$ of even order may be restricted to transversals whose supports in D do not contain odd cycles. □

Exercises

2.1 *Give a proof of the statement that the determinant of an odd oder skew-symmetric matrix is zero, using Proposition 2.5.*

2.2 *Let D be an orientation of the Petersen graph with positive weights attached to all its arcs, and let $\mathbf{A}$ be the skew-symmetric matrix represented by D. How many transversals of $\mathbf{A}$ of nonzero weight are there which do not include odd cycles?*

3 Perfect Matchings Enter the Game

Let $\mathbf{A}$ be an $n \times n$ skew-symmetric matrix, where n is even. By Theorem 2.6, the expansion of the determinant of $\mathbf{A}$ may be restricted to those transversals of nonzero weight whose supports in D are disjoint unions of arcs (representing flattened directed cycles of length two) and even cycles with assigned senses of traversal. Henceforth, by a transversal of $\mathbf{A}$ we shall mean one which satisfies this property.

Suppose that T is a transversal of $\mathbf{A}$ and $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_\ell$ is the support of T, where each γ_i is either a single arc of D representing a flattened directed cycle of length two; or is an even cycle (not necessarily a directed cycle) of length four or more in D with a prescribed sense of traversal. If all γ_i are arcs, then the set of edges of G which correspond to that set of arcs in D which belong the support of T is a perfect matching in G. Alternatively, suppose that, adjusting notation, $\gamma_1, \gamma_2, \ldots, \gamma_k$ are even cycles of length four or more in D with prescribed senses of traversal, for some $k \leq \ell$; and $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_\ell$ are arcs of D representing directed cycles of length two. If we disregard the orientations of the arcs in the support of T, we would have a spanning subgraph of G whose components are either even cycles or copies of K_2. Thus the set of edges of G which corresponds to the set of arcs in the support of the transversal of T may be expressed as the union of two perfect matchings of G; in some cases, one may be able to do this in more than one way. We give three examples below to illustrate how these cases arise. All these examples pertain to the matrix $\mathbf{A}$ represented by Figure 1.

Example 3.1 *Let $T := \{(1,t,1), (2,9,2), (8,3,8), (7,4,7), (5,6,5)\}$ be a transversal whose support consists entirely of cycles of length two. It corresponds to the set $\{(1,t), (2,9), (8,3), (7,4), (5,6)\}$ of arcs of D and, in turn, the corresponding set $M := \{1t, 29, 83, 74, 56\}$ of edges of G is a perfect matching of G.*

The weight of T is $(-a_{1t}^2).(-a_{29}^2).(-a_{83}^2).(-a_{74}^2).(-a_{56}^2) = -a_{1t}^2.a_{29}^2.a_{83}^2.a_{74}^2.a_{56}^2$; and the sign of T is minus because it is the union of an odd number of even cycles. Therefore $\mathrm{sign}(T)\mathrm{wt}(T) = a_{1t}^2.a_{29}^2.a_{83}^2.a_{74}^2.a_{56}^2$.

Example 3.2 *Let γ_1 be the 8-cycle $(5,6,7,8,9,2,3,4,5)$, and let $\gamma_2 := \{(1,t,1)\}$. Then, there are two transversals T_1 and T_2 whose supports in D are, respectively, $\gamma_1^c \cup \gamma_2$ and $\gamma_1^a \cup \gamma_2$. (The cycles γ_1^c and γ_1^a in D correspond, respectively, to the directed cycles $(5,6,7,8,9,2,3,4,5)$ and $(5,4,3,2,9,8,7,6,5)$ in D^*; and the arc $(1,t)$ of D represents the directed cycle $(1,t,1)$ in D^*.) Thus, for $i = 1,2$, the spanning subgraph of G which is obtained by disregarding the orientations of arcs in the support of T_i has two components, one of which is the 8-cycle*

$(5,6,7,8,9,2,3,4,5)$ *and the other has just one edge, namely* $1t$. *The edge set of this subgraph of* G *is the union of the two perfect matchings* $M := \{56, 78, 92, 34, 1t\}$ *and* $N := \{67, 89, 23, 45, 1t\}$.

For $i = 1, 2$, the support of T_i has two components, both of which are even cycles. Therefore both $\mathrm{sign}(T_1)$ and $\mathrm{sign}(T_2)$ are plus. Furthermore, by Corollary 2.4, $\mathrm{w}t(T_1) = \mathrm{w}t(T_2)$. Therefore $\mathrm{sign}(T_1)\mathrm{w}t(T_1) = \mathrm{sign}(T_2)\mathrm{w}t(T_2)$.

EXAMPLE 3.3 *Let* γ_1 *and* γ_2 *denote, respectively, the two cycles* $(1,5,6,t,9,2,1)$ *and* $(7,8,3,4,7)$ *in* D. *There are four transversals whose support in* D *is the union of the arcs sets of* γ_1 *and* γ_2, *namely (i)* T_1 *whose support in* D *is* $\gamma_1^c \cup \gamma_2^c$; *(ii)* T_2 *whose support in* D *is* $\gamma_1^c \cup \gamma_2^a$; *(iii)* T_3 *whose support in* D *is* $\gamma_1^a \cup \gamma_2^c$; *and (iv)* T_4 *whose support is* $\gamma_1^a \cup \gamma_2^a$. *For* $1 \le i \le 4$, *the subgraph of* G *which is obtained by disregarding the orientations of the arcs of the support of* T_i *is the union of the two disjoint even cycles* $(1,5,6,t,9,2,1)$ *and* $(7,8,3,4,7)$. *Its edge set may be expressed as the union of two perfect matchings in two different ways, namely:* $\{15, 6t, 29, 78, 34\} \cup \{56, t9, 21, 83, 24\}$*; and also as* $\{15, 6t, 29, 83, 47\} \cup \{56, t9, 21, 78, 34\}$.

By Corollary 2.4, all T_1, T_2, T_3 and T_4 have the same weight and, since their supports are unions of two even cycles, their signs are positive. Thus, $\mathrm{sign}(T_i)\mathrm{w}t(T_i) = \mathrm{sign}(T_j)\mathrm{w}t(T_j)$, for $1 \le i \le j \le 4$.

In the next section, we shall elaborate on the role perfect matchings play in the theory of determinants of even order skew-symmetric matrices.

4 Weights and Signs of Perfect Matchings

4.1 Weights of perfect matchings

Suppose that $\mathbf{A}$ is a skew-symmetric matrix of even order and that T is a transversal of $\mathbf{A}$ whose support in D is the **disjoint union** of $\gamma_1, \gamma_2, \ldots, \gamma_k$, where each γ_i is either an arc or an even cycle in D with a prescribed sense of traversal. For convenience, we shall assume that, for all cycles of length four or more, this sense is 'clockwise'. By Corollary 2.4, this incurs no loss of generality!

For $1 \le i \le k$, let C_i be the subgraph of G obtained by disregarding the orientations of arcs in γ_i. Then either C_i is a copy of K_2 or is an even cycle, and the edge set of $C_1 \cup C_2 \cup \cdots \cup C_k$ may be expressed as the union of two, not necessarily distinct, perfect matchings, say M and N, of G. Furthermore, for $1 \le i \le k$, $M_i := E(C_i) \cap M$ and $N_i := E(C_i) \cap N$ are perfect matchings of C_i, and $M_i \cap N_i = \emptyset$ if C_i is an even cycle, and $M_i = N_i$ if C_i is a copy of K_2.

The *weight of* C_i *in* G, for $1 \le i \le k$, denoted by $\mathrm{w}t(C_i)$, is the product of the weights of edges of C_i, and weight of $\mathrm{w}t(C_1 \cup C_2 \cup \cdots \cup C_k)$ in G is the product of the weights of $C_1, C_2, \ldots, C_k$. Since weights of all edges of G are positive, this weight is positive. However, the weight of $\gamma_1^c \cup \gamma_2^c \cup \cdots \cup \gamma_k^c$, which is relevant to the computation of the determinant of $\mathbf{A}$, depends on the orientations of the arcs in D. For $1 \le i \le k$, the weight of γ_i^c in D is the same as the weight of C_i in G, if the number of reverse arcs in γ_i^c is even, and it is the negative of the weight of C_i in G, if the number of reverse arcs in γ_i^c is odd. But the number of reverse arcs in γ_i^c is the sum of the number of reverse arcs in the perfect matching M_i and those in the perfect matching N_i.

Taking the above observations into account, the key idea now is to define, for each perfect matching M of G, the notions of $\mathrm{sign}(M)$ weight $\mathrm{w}t(M)$ of M and show that, given any transversal T of $\mathbf{A}$, there exist two perfect matchings M and N (which maybe the same) such that:

$$\mathrm{sign}(T) \cdot \mathrm{w}t(T) = (\mathrm{sign}(M) \cdot \mathrm{w}t(M)) \ \cdot \ (\mathrm{sign}(N) \cdot \mathrm{w}t(N)) \tag{1}$$

4.2 Weights of perfect matchings

The *weight of a perfect matching* M of G, denoted by $\mathrm{w}t(M)$, is the product of the weights of edges of M in G. Since the weights of all edges in G are positive, the weights of all perfect matchings are positive.

We now proceed to define the concept of the sign of a perfect matching of G which takes into account the orientations of its edges in the digraph D. This necessarily involves the notion of a permutation of the vertex set of G which we have avoided so far. To prove Cayley's Theorem, we shall also need a few basic facts from the theory of permutations of a finite set. A review of the definitions, terminology and the simple results related to permutations we use can be found in the appendix at the end.

4.3 Signs of perfect matchings

Permutations associated with perfect matchings and their signs

Let $M = \{e_1, e_2, \ldots, e_r\}$ be a perfect matching of G, and for $1 \leq i \leq r$, let u_i and v_i denote, respectively, the tail and the head of e_i in D. Then the permutation $\pi(M)$ associated with M is:

$$\pi(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & . & . & . & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & . & . & . & u_r & v_r \end{pmatrix}$$

Signs of perfect matchings: The *sign* of M of G (with respect to the digraph D), denoted by $\mathrm{sign}(M)$, is the sign of the permutation $\pi(M)$.

Example 4.1 *Consider the perfect matchings $M := \{12, 9t, 65, 78, 34\}$ and $N := \{29, t6, 51, 83, 47\}$ in the underlying undirected graph of the digraph D shown in Figure 1. Then,*

$$\pi(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & t \\ 2 & 1 & 9 & t & 5 & 6 & 7 & 8 & 4 & 3 \end{pmatrix}$$

$$\pi(N) := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & t \\ 2 & 9 & 6 & t & 5 & 1 & 8 & 3 & 7 & 4 \end{pmatrix}$$

It can be verified that $\pi(M)$ decomposes into four odd cyclic permutations and two even cyclic permutations, and hence, $\mathrm{sign}(M)$ *is plus; on the other hand, $\pi(N)$*

decomposes into two odd cyclic permutations and one even cyclic permutations, and hence sign(N) *is minus. (One might wonder if signs of M and depend on the order in which the edges of M are enumerated. They do not. (Exercise 4.2).)*

The following proposition should give the reader an intuitive idea of the meaning of the signs of perfect matchings. (To understand this statement, try to verify the result for two or three different orientations of $C_8 = (1, 2, \dots, 8, 1)$.)

Proposition 4.2 *Let G be the even cycle $(1, 2, \dots, 2n, 1)$ of length four or more, let M and N be the two perfect matchings of G, and let $D = \gamma$ be an orientation of G. Let M be the perfect matching containing the edge* 12 *and N be the perfect matching containing the edge* 23. *Suppose that we assign a clockwise sense of traversal to the cycle γ. Then*

(i) *the sign of M with respect to γ^c is* plus *if the number of reverse arcs that belong to M is* even, *and is* minus *if that number is* odd*;*

(ii) *in contrast to the sign of M, the sign of N with respect to γ^c is* minus *if the number of reverse arcs that belong to N is* even, *and is* plus *if that number is* odd.

(iii) *The two perfect matchings M and N of G have the same signs with respect to γ^c if the number of reverse arcs in γ^c is odd, and different signs if that number is even. (This follows from items (i) and (ii) and the fact that the number of reverse arcs in γ^c is the sum of the numbers of reverse arcs that belong M and of those which belong to N). As the parity of the number of reverse arcs in γ^a is the same as the parity of the number of reverse arcs in γ^c, the analogous result also holds with respect to γ^a.* □

The third item in the statement of the above proposition holds even if the vertices of γ^c are not labeled in the cyclic order $(1, 2, \dots, n)$. That is the essence of the following statement:

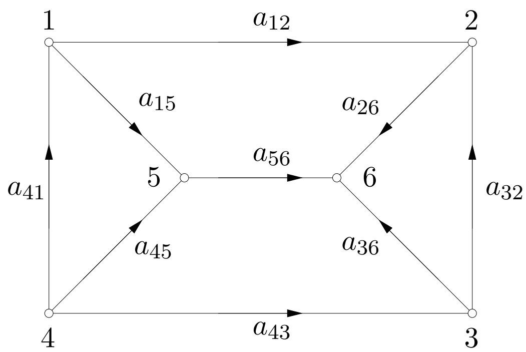
A crucial lemma

Lemma 4.3 *Let C be an even cycle of length four or more and let M and N denote the two perfect matchings of C. Consider any orientation γ of the cycle C. Then,* sign(M) = sign(N) *(or, equivalently,* sign$(M) \cdot 1$sign(N) *is positive) in the digraph γ if and only if the parity of the number of reverse arcs in any sense of traversal of γ is odd.* □

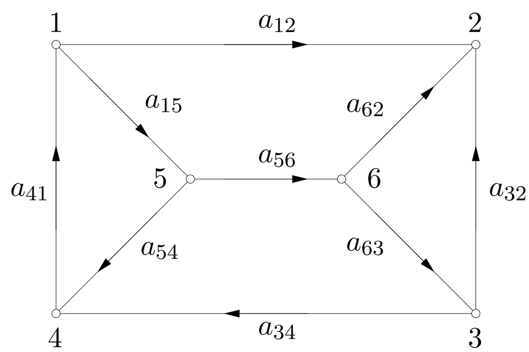
The result above, which is stated differently in our book and appears as Exercise 19.2.1, plays a crucial role in the proof of Cayley's Theorem. Solution to 19.2.1 is given in Appendix A of the book, but try to find a proof of Lemma 4.3 which is your own.

Exercises

4.1 *Figure 2 depicts two different orientations D_1 and D_2 of $\overline{C_6}$. Graph G, the underlying undirected graph of both D_1 and D_2, has has four perfect matchings, namely $M_1 := \{12, 36, 45\}$, $M_2 := \{15, 34, 26\}$, $M_3 := \{14, 23, 56\}$, and $M_4 := \{12, 34, 56\}$. Determine the signs of these four perfect matchings with respect to each of D_1 and D_2.*



(a) D_1



(b) D_2

Figure 2: Two different orientations of C_6.

4.2

(i) *Verify that the sign of the perfect matching $M := \{15, 26, 34\}$ of the digraph D_1 shown in Figure 2 is plus regardless of the order in which the edges of M are enumerated.*

(ii) *Prove the general fact that the sign of a perfect matching M of G does not depend on the order in which the edges of M are enumerated.*

4.3 *Give your own proof of Lemma 4.3.*

5 Pfaffians and Cayley's Theorem

In the abstract we indicated that the determinant of a skew-symmetric matrix $\mathbf{A}$ of even order is the square of a certain polynomial in the entries of $\mathbf{A}$. We now have all the terminology needed to define that polynomial.

5.1 The Pfaffian

Let $\mathbf{A}$ be a skew-symmetric matrix of even order, let D be the simple digraph representing $\mathbf{A}$, and let G be the underlying undirected graph of D. Them, the *Pfaffian* of $\mathbf{A}$, denoted by $\mathrm{Pf}(\mathbf{A})$ is

$$\mathrm{Pf}(\mathbf{A}) := \sum_{M \in \mathcal{M}(G)} \mathrm{sign}(M)\mathrm{wt}(M) \tag{2}$$

where $\mathcal{M}(G)$ denotes the set of all perfect matchings of G.

EXAMPLE 5.1 *Consider the two orientations D_1 and D_2 of $G := \overline{C_6}$ depicted in Figure 2. The undirected graph G, as we noted, has four perfect matchings $M_1 = \{12, 36, 45\}$, $M_2 = \{15, 34, 26\}$, $M_3 = \{14, 23, 56\}$ and $M_4 = \{12, 34, 56\}$. The weights of these four perfect matchings are, respectively,*

$$a_{12}a_{36}a_{45},\ a_{15}a_{34}a_{26},\ a_{14}a_{23}a_{56},\ \text{and } a_{12}a_{34}a_{56}.$$

The signs of M_1 and M_3 in D_1 are plus, where as those of M_2 and M_4 are minus (see Exercise 4.1.) Therefore, in this case:

$$\mathrm{Pf}(\mathbf{A}) = a_{12}a_{36}a_{45} - a_{15}a_{34}a_{26} + a_{14}a_{23}a_{56} - a_{12}a_{34}a_{56}$$

On the other hand, with respect to the orientation orientation D_2, the signs of all the four perfect matchings are plus, and their weights are, respectively,

$$a_{12}a_{63}a_{54},\ a_{15}a_{34}a_{62},\ a_{41}a_{32}a_{56},\ \text{and } a_{12}a_{34}a_{56}.$$

Therefore, in this case:

$$\mathrm{Pf}(\mathbf{A}) = a_{12}a_{63}a_{54} + a_{15}a_{34}a_{62} + a_{41}a_{32}a_{56} + a_{12}a_{34}a_{56}.$$

Observe that all perfect matchings of $\overline{C_6}$ have the same sign with respect to D_2 which is not the case with respect to D_1. In the language of Chapter 19, digraph D_2 is a Pfaffian orientation of G and D_1 is not!

5.2 Cayley's Theorem

THEOREM 5.2 *The determinant of an even order skew-symmetric matrix $\mathbf{A}$ is the square of its Pfaffian.*

Proof: In the definition of the Pfaffian of a digraph D (Equation (2)) we used $\mathbf{M}$ as a generic symbol for a perfect matching of G. In the expression for the square of the Pfaffian of D, we shall use $\mathbf{M}$ as the generic symbol in one factor and $\mathbf{N}$ as the generic symbol in the other. With this understanding, the square of the Pfaffian of D is the following expression:

$$\begin{aligned}(\mathrm{Pf}(\mathbf{A}))^2 &= \sum_{M\in\mathcal{M}(G)} \mathrm{sign}(M)\mathrm{wt}(M) \sum_{N\in\mathcal{M}(G)} \mathrm{sign}(N)\mathrm{wt}(N)\\ &= \sum_{M\in\mathcal{M}(G)}\ \sum_{N\in\mathcal{M}(G)} \mathrm{sign}(M)\mathrm{wt}(M)\mathrm{sign}(N)\mathrm{wt}(N).\end{aligned}$$

Observe that if M and N are two distinct perfect matchings of G, then there are two terms in the above expression. One of these corresponds to the ordered pair (M, N) and the other to (N, M). But if $M = N$ there is just one term, namely $(\mathrm{sign}(M).\mathrm{wt}(M))^2$.

What needs to proved?

Let $\mathbf{A}$ be a skew-symmetric matrix of order $2r$. Recall that the determinant $\det(\mathbf{A})$ of $\mathbf{A}$ is $\sum_T \operatorname{sign}(T)\mathrm{w}t(T)$, where the sum is taken over the set of all transversals T of $\mathbf{A}$. Since the order of $\mathbf{A}$ is even, we need consider only those transversals T of nonzero weight whose supports do not include any odd cycles. Thus, to prove the validity of the statement of Cayley's Theorem, we need to establish a one-to-one correspondence between the set of all such transversals of $\mathbf{A}$ and the set of ordered pairs (M, N) of perfect matchings of G (allowing for the possibility that $M = N$) such that:

$$\operatorname{sign}(T)\mathrm{w}t(T) = \operatorname{sign}(M)\mathrm{w}t(M) \cdot \operatorname{sign}(N)\mathrm{w}t(N) \tag{3}$$

5.3 Equivalence classes of transversals and ordered pairs of perfect matchings

Our first task is to establish a one-to-one correspondence between the set of all transversals of $\mathbf{A}$ and ordered pairs of perfect matchings of G. Thus, let T be a transversal of $\mathbf{A}$ whose support in D has ℓ cycles $\gamma_1, \gamma_2, \ldots, \gamma_\ell$, of which the first k cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ are of length four or more, each with an assigned sense of traversal, and the remaining cycles $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_\ell$ are flattened directed cycles of length two represented by arcs in D.

Let us first consider the case in which $k = 0$. In this case then $\ell = r$ and the set of edges of G which correspond to the arcs of D representing the flattened cycles $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ is a perfect matching of G.

Now supose that $k \geq 1$. In this case, each of the k cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ may, independently, be assigned either the clockwise or the anti-clockwsie sense of traversal. Thus T belongs to a class $\mathcal{T}$ of 2^k transversals of $\mathbf{A}$. (By Corollary 2.4, any two members of $\mathcal{T}$ have the same weight and the same sign.)

On the other hand, by disregarding the orientations of the arcs of D in the support of T, we obtain a subgraph H of G which is a union of k disjoint even cycles $C_1, C_2, \ldots, C_k$ corresponding the cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ of D; and copies of K_2 corresponding to the arcs of D representing the flattened cycles $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_\ell$. Thus the edge set $E(H)$ of H may be expressed as a union of two perfect matchings; in fact $E(H)$ may be expressed as the union of two perfect matchings in $2^{(k-1)}$ different ways. Thus, there are 2^k ordered pairs (M, N) of perfect matchings of the graph G such $E(H) = M \cup N$.

It is therefore possible to establish a one-to-one correspondence between the set $\mathcal{T}$ of transversals and ordered pairs of perfect matchings whose union is $E(H)$. One 'natural' way to do this is as follows: Colour the edges of each component of H that is an even cycle, alternately, in two colours c (crimson) and a (amber), and colour the edges of H corresponding flattened cycles of length two by a neutral colour b (beige) which is meant to be both c and a. In each of the components $C_1, C_2, \ldots, C_k$ of H which are even cycles, the set of edges labelled c and those labelled a are disjoint perfect matchings of that component. Any combination of these matchings could be extended to a perfect matching of H by including those edges of H which are coloured by the neutral colour b.

Given any transversal T in $\mathcal{T}$, there is another member T' of $\mathcal{T}$, which we shall refer to as the *complement* of T, that is obtained by reversing the senses of traversal assigned to the cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$. It can be verified that if the ordered pair of perfect matchings associated with T is (M, N), the the ordered pair associated with its complement T' is (N, M). For a concrete example, see Exercise 5.1

5.4 Proof of Cayley's Theorem

For notational convenience, we split the proof into two cases, the first being the one that is easier to deal with.

CASE 1 *Let T be a transversal of $\mathbf{A}$ whose support in D is a subdigraph of D whose arc set is disjoint union of r arcs, say $\{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$.*

For $1 \leq i \leq r$, the arc (u_i, v_i) of D corresponds to the directed cycle (u_i, v_i, u_i) in D^*. Since $\mathbf{A}$ is skew-symmetric, $\mathrm{wt}(u_i, v_i, u_i)$ is $-(a_{u_iv_i})^2$, and its sign is minus because it is an even cycle. Therefore

$$\mathrm{sign}(T)\mathrm{wt}(T) = (a_{u_1v_1}a_{u_2v_2}\ldots a_{u_rv_r})^2.$$

Clearly $M := \{u_1v_1, u_2v_2, \ldots, u_rv_r\}$ is a perfect matching of G. Regardless of what the sign of M in D is,

$$\mathrm{sign}(T)\mathrm{wt}(T) = (a_{u_1v_1}a_{u_2v_2}\ldots a_{u_rv_r})^2 = (\mathrm{sign}(M)\mathrm{wt}(M))(\mathrm{sign}(M)\mathrm{wt}(M))$$

Thus, in this case, the required equality holds. $\diamond$

CASE 2 *Let T be a transversal of the skew-symmetric matrix $\mathbf{A}$ of even order, whose support in D has k even cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ of length four or more, where $k \geq 1$, each with an assigned sense of traversal; and flattened cycles $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_\ell$ of length two.*

Let $\mathcal{T}$ denote the set of all transversals of $\mathbf{A}$ whose support in D is the union of $\gamma_1, \gamma_2, \ldots, \gamma_\ell$; and consider any bijection between $\mathcal{T}$ and the set of all ordered pairs of perfect matchings of the subgraph H of G obtained by disregarding the orientations of the arcs in D. Now, let (M, N) be the pair of perfect matchings of H associated with the given transversal T, and for $1 \leq i \leq \ell$, let $M_i := M \cap E(C_i)$ and $N_i := N \cap E(C_i)$. Observe that, for $1 \leq i \leq k$, the two subsets M_i and N_i of $E(H)$ are disjoint perfect matchings of C_i such that $M_i \cup N_i = E(C_i)$; and, for $k+1 \leq i \leq \ell$, $M_i = N_i = E(C_i)$.

Clearly $\mathrm{wt}(M) \cdot \mathrm{wt}(N) = \mathrm{wt}(M_1) \cdot \mathrm{wt}(N_1) \cdot \mathrm{wt}(M_2) \cdot \mathrm{wt}(N_2) \cdot \ldots \cdot \mathrm{wt}(M_k) \cdot \mathrm{wt}(N_k)$ does not depend on the labeling of the vertices of D. **The signs $\mathrm{sign}(M)$ and $\mathrm{sign}(N)$ of perfect matchings M and N of G do depend on the labelings of the vertices of D. However, the product $\mathrm{sign}(M)\mathrm{sign}(N)$ does not. This follows from Lemma 4.3.** With this understanding, let us relabel the vertices in such a way that, for $1 \leq i \leq k$, the labels of vertices of C_i is a segment of $|V(C_i)|$ consecutive integers of the vertex set $\{1, 2, \ldots, 2r\}$ of G; and let $\mathrm{sign}(M_i)$ and $\mathrm{sign}(N_i)$ denote respectively, the signs of M_i and N_i in the subdigraph γ_i of D; and note that

$$\mathrm{sign}(M) = \mathrm{sign}(M_1) \cdot \mathrm{sign}(M_2) \cdot \ldots \cdot \mathrm{sign}(M_\ell) \tag{4}$$

$$\mathrm{sign}(N) = \mathrm{sign}(N_1) \cdot \mathrm{sign}(N_2) \cdot \ldots \cdot \mathrm{sign}(N_\ell). \tag{5}$$

Thus, to establish the desired Equation 3, it suffices to show that, for $i = 1, 2, \ldots, \ell$,

$$\mathrm{sign}(\gamma_i)\mathrm{w}t(\gamma_i) = \mathrm{sign}(M_i)\mathrm{w}t(M_i) \cdot \mathrm{sign}(N_i)\mathrm{w}t(N_i). \tag{6}$$

For $1 \leq i \leq \ell$, the cycle γ_i is even, implying that $\mathrm{sign}(\gamma_i) = -1$. (By Corollary 2.4, $\mathrm{w}t(\gamma_i^c) = \mathrm{w}t(\gamma_i^a)$. We refer to this common value as $\mathrm{w}t(\gamma_i)$.) The cycles $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_\ell$ are of length two, say $\gamma_i = (u_i, v_i, u_i)$. Then, as in Case 1, $\mathrm{sign}(\gamma_i)\mathrm{w}t(\gamma_i) = a_{u_iv_i}^2$. Furthermore, $M_i = N_i$, and $\mathrm{w}t(M_i) = \mathrm{w}t(N_i) = a_{u_iv_i}$. Hence Equation (6) holds for $k + 1 \leq i \leq \ell$.

In the remaining cases, that is, for $1 \leq i \leq k$, we need subtler arguments to establish the validity of Equation (6). Since $M_i \cup N_i = E(C_i)$, it follows that $\mathrm{w}t(M_i).\mathrm{w}t(N_i) = \mathrm{w}t(C_i)$, Equation (6) may be rephrased as:

$$\mathrm{sign}(\gamma_i)\mathrm{w}t(\gamma_i) = (\mathrm{sign}(M_i)\mathrm{sign}(N_i)).\mathrm{w}t(C_i), \tag{7}$$

where $\mathrm{w}t(C_i)$ denotes the product of the weights of edges of the cycles C_i which does not depend on the orientations of the corresponding arcs in D.

We now consider two subcases depending on the parity of the number of reverse arcs in γ_i with respect to the orientation D. (Note that this number depends on the sense of traversal assigned to γ_i, and the senses of traversal assigned to constituent cycles is a part of the description of a transversal.)

Case 2.1 *The number of reverse arcs in γ_i is even.*

In this case, $\mathrm{w}t(\gamma_i)$ is equal $\mathrm{w}t(C_i)$ which is positive because the weights of all edges of G are positive. On the other hand, by Lemma 4.3, $\mathrm{sign}(M_i) \neq \mathrm{sign}(N_i)$, implying that $\mathrm{sign}(M_i)\mathrm{sign}(N_i) = -1$. Hence Equation (7) holds.

Case 2.2 *The number of reverse arcs in γ_i is odd.*

In this case, $\mathrm{w}t(\gamma_i)$ is equal $-\mathrm{w}t(C_i)$ and, as noted earlier, $\mathrm{sign}(\gamma_i) = -1$. On the other hand, by Lemma 4.3, $\mathrm{sign}(M_i) = \mathrm{sign}(N_i)$, implying that $\mathrm{sign}(M_i)\mathrm{sign}(N_i) = +1$. Thus, Equation (7) holds in this case as well! □

Exercise

5.1 *Let* $\mathbf{A}$ *be a skew-symmetric matrix of order sixteen, and let* $\mathcal{T}$ *denote the set of all transversals of* $\mathbf{A}$ *whose support in* D *has three cycles* γ_1, γ_2 *and* γ_3 *of lengths four, six and four, respectively, and one flattened cycle* γ_4 *of length two, and suppose that the octagonal prism shown in Figure 3 is the corresponding subgraph* H *of* G *obtained by disregarding the orientations of the arcs in* D.

Find the ordered pairs of perfect matchings associated with the eight possible transversals with the given profile.

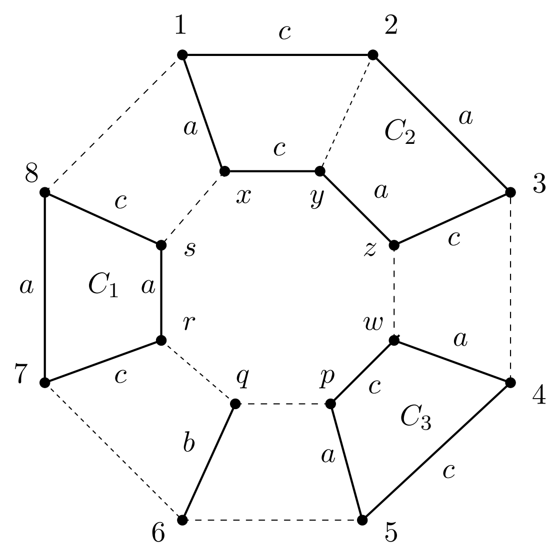


Figure 3: The support in G of a transversal T in $\mathcal{T}$.

Appendix

A permutation of a set S is a bijection of S onto itself. When S is a finite set, permutations are usually specified using the familiar two-line notation. (If $S = \{1, 2, \ldots, n\}$, a permutation π of S is specified by listing $(1, 2, \ldots, n)$ in the first line and $(\pi(1), \pi(2), \ldots, \pi(n))$ in the second line so that $\pi(i)$ appears below i, for $1 \leq i \leq n$.)

Here we recall the basic facts related to permutations of finite sets using the terminology we have used used in sections 3 and 4.

6 Cyclic Permutations

Let $S := (i_1, i_2, \ldots, i_k)$ of k be an <u>cyclically ordered</u> set consisting of k distinct indices. Then the permutation

$$\gamma = \begin{pmatrix} i_1 & i_2 & \ldots & i_{k-1} & i_k \\ i_2 & i_3 & \ldots & i_k & i_1 \end{pmatrix} \tag{8}$$

which permutes the indices of S cyclically is called a *cyclic permutation* of S and is denoted simply by $(i_1 \;\, i_2 \;\, \ldots \;\, i_k)$. (Thus, we use one-line notation for cyclic permutations.)

By reversing the order of elements in S, we obtain the cyclically ordered set $(i_1, i_k, \ldots, i_2)$. The corresponding cyclic permutation is

$$\delta = \begin{pmatrix} i_1 & i_k & \ldots & i_3 & i_2 \\ i_k & i_{k-1} & \ldots & i_2 & i_1 \end{pmatrix} \tag{9}$$

In one-line notation, δ is $(i_k \; i_{k-1} \; \ldots \; i_1)$.

Example 6.1 *The permutation γ which maps 3 to 2; 2 to 4; 4 to 1; and 1 back to 3 is the cyclic permutation* $(3\ 2\ 4\ 1)$*; and the permutation δ which maps 3 to 1, 1 to 4, 4 to 2, and 2 back to 3 is the cyclic permutation* $(3\ 1\ 4\ 2)$.

Notice that δ 'undoes' what σ 'does' in the sense that if γ maps an element i of the index set to an element j, then δ maps j back to i. Similarly, γ 'undoes' what δ 'does'. For this reason, we refer to γ and δ as *inverses* of each other and refer to δ as γ^{-1} and γ as δ^{-1}.

The notion of cyclic permutations will become clearer when we start using the language of graphs which will be introduced in a later section.

6.1 Decomposition into cyclic permutations

Any permutation π of a set $S := \{1, 2, \ldots, n\}$ decomposes uniquely into cyclic permutations of disjoint subsets of S. To illustrate what this statement means, let us consider the permution π of $\{1, 2, \ldots, 10\}$ shown below:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 6 & 8 & 7 & 1 & 5 & 2 & 10 & 9 \end{pmatrix} \tag{10}$$

This permutation π sends (maps) 1 to 3, 3 to 6 and 6 back to 1; sends 2 to 4, 4 to 8, and 8 back to 2; sends 5 to 7, and 7 back to 5; and finally, 9 to 10 and 10 back to 9. Thus, π decomposes into the four cyclic permutations (1 3 6), (2 4 8), (5 7) and (9 10).

It is common practice to present a permutation in terms of its decomposition into cyclic permutations. Thus,

$$\pi := (1\ 3\ 6)(2\ 4\ 8)(5\ 7)(9\ 10),$$

where π is the permutation displayed in 10. Note that π decomposes into two odd cyclic permutations and two even cyclic permutations.

We sometimes refer to the cyclic permutations into which a permutation π decomposes as its 'factors' and π as the 'product' of those factors.

6.2 Signs of permutations

Any permutation π of a set $\{1, 2, \ldots, n\}$ maybe transformed into the identity permutation by means of a sequence of transpositions. There may clearly be different ways of achieving this. An important result in combinatorics says that the parity of the number transpositions required to transform π into the identity permutation is always the same. If this parity is even, then π is an *even* permutation, and if it is odd, then π is an *odd*. The sign of π, denoted by $\text{sign}(\pi)$ is $+$ (or $+1$) if π is even, and $-$ (or -1) if it is odd. (Suppose that π and ρ are two permutations of $\{1, 2, \ldots, n\}$. Then the number of transpositions required to transform π to ρ is even if and only if they have the same parity.)

Proposition 6.2 The sign of an even cyclic permutation is minus, whereas the sign of an odd cyclic permutation is plus. □

Proposition 6.3 *The sign of a permutation π of $(1, 2, \ldots, n)$ is plus if the number of even cycles in the decomposition of π into cyclic permutations is even, and is minus if that number is odd. (Thus, for example, the sign of the permutation π displayed in equation (10) is plus.)* □

Of the six permutations of $S = \{1, 2, 3\}$, the signs of the three permutations (1)(2)(3), (1 2 3) and (1 3 2) are even; and those of (1)(2 3), (2)(1 3) and (3)(1 2) are odd.

Exercises

6.1 *Find a permutation of $\{1, 2, \ldots, 10\}$ which decomposes into two odd cyclic permutations and one even cyclic permutation.*

6.2 *Of the 24 permutations of $\{1, 2, 3, 4\}$, there are 12 that are even and 12 that are odd. List all of them.*