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Isometric immersions of space forms into Riemannian and pseudo-Riemannian spaces of constant curvature

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Abstract. This survey contains results on local and global isometric immersions of two-dimensional and multidimensional Riemannian and pseudo-Riemannian space forms into spaces of constant sectional curvature.

Contents

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Introduction

There is a vast literature devoted to isometric immersion of multidimensional Riemannian manifolds into Euclidean spaces, including the surveys by Gromov and Rokhlin [80], Aminov [10], Jacobowitz [94], and Poznyak and Sokolov [138].

If the codimension of an immersion is large, then there is a lot of freedom, and no close relationship between the intrinsic and extrinsic geometry of the manifold. However, stringent connections appear between the extrinsic geometry of the manifold and its curvature under isometric immersion of spaces of constant sectional curvature with sufficiently small codimension of an embedding. Some problems in this area are still unsolved. In particular, the problem of multidimensional generalization of the Hilbert theorem remains open.

This survey is an exposition of local and global results on isometric immersions of Riemannian and pseudo-Riemannian manifolds of constant sectional curvature.

At the 1987 Novosibirsk conference dedicated to the 75th birthday of Academician A. D. Aleksandrov, D. V. Alekseevskii asked me several questions on this topic, and suggested writing a survey. I have finally realized this intention and now cordially thank Alekseevskii for the idea. I am also indebted to the referee for comments.

CHAPTER 1

ISOMETRIC IMMERSIONS OF MULTIDIMENSIONAL RIEMANNIAN MANIFOLDS

1.1. Local and global results. In differential topology a map of a smooth manifold into another smooth manifold is called an *embedding* if it is a diffeomorphism onto a submanifold and an *immersion* if it is locally an embedding, that is, it has a non-singular differential (and is sufficiently smooth). In Riemannian geometry such maps are also assumed to be isometric.

Any smooth manifold of dimension n admits a differentiable embedding in E^{2n}, and if n is not a power of two, then in E^{2n-1} as well. For any n every smooth manifold of dimension n without closed components admits a differentiable embedding in E^{2n-1}, and for any $n > 1$ every smooth manifold of dimension n admits a differentiable immersion into E^{2n-1}. If n is a power of two, then there are smooth closed manifolds of dimension n that admit neither differentiable nor topological embeddings in E^{2n-1} and cannot be immersed into E^{2n-2}; for instance, the real projective space $\mathbb{R}P^n$ is such a space.

The following theorem was proved for isometric immersions.

Theorem 1.1.1 (Cartan [45], Janet [96]). *In any analytic Riemannian manifold of dimension n with a distinguished point there is a neighbourhood of the distinguished point that admits an isometric analytic embedding in* E^{s_n}, *where* $s_n = \frac{1}{2}n(n+1)$, *and one cannot replace* s_n *by* $s_n - 1$.

For a C^2 map f of a smooth C^2 manifold V of dimension n into E^q we denote by $T_v^2(f)$ the osculating space spanned by the first and second derivatives of the radius vector $x = x(u)$:

$$\frac{\partial x}{\partial u_i} \quad (i = 1, \dots, n), \qquad \frac{\partial^2 x}{\partial u_i \, \partial u_j} \quad (i, j = 1, \dots, n),$$

where $u = (u_1, \dots, u_n)$ are the coordinates on V. If $\dim T_v^2(f) = n + s_n$, then the map is said to be *free*.

Theorem 1.1.2 (Gromov and Rokhlin [80]). *In any Riemannian manifold of dimension n and class* C^∞ *with a distinguished point there is a neighbourhood of the distinguished point that admits a free isometric embedding in* E^{s_n+n}.

Theorem 1.1.3 (Jacobowitz [94]). *Any* C^l *regular Riemannian manifold locally admits a* C^l *regular isometric embedding in the Euclidean space* E^{s_n+n}, *where l is an arbitrary number with* $2 < l \leqslant \infty$.

Under weaker regularity conditions, some neighbourhood of an arbitrary point of any C^1 Riemannian n-dimensional manifold can be C^1 isometrically embedded in the Euclidean space E^{n+1} [107], [122].

The following theorems hold for global isometric immersions.

Theorem 1.1.4 (Nash, see [117]). *Every closed Riemannian manifold of dimension n and class* C^r $(3 \leqslant r \leqslant \infty)$ *admits an isometric* C^r *embedding in* E^q, *where* $q = 3s_n + 4n = \frac{1}{2}(3n^2 + 11n)$.

Theorem 1.1.5 (Gromov and Rokhlin, see [117]). *A compact Riemannian manifold of class* C^∞ *admits a* C^∞ *isometric free embedding in* E^q, *where* $q = s_n + 3n + 5$.

Stronger results hold for two-dimensional Riemannian manifolds. Any compact two-dimensional Riemannian manifold of class C^a (C^∞) admits a C^a (C^∞) embedding in E^{10} if it has no closed components; a Riemannian manifold diffeomorphic to a sphere can be isometrically embedded in E^7 [80].

A continuous map is said to be *proper* if the pre-images of compact sets are compact. If a map is a topological embedding, then this condition holds if and only if the image is closed.

If a Riemannian manifold admits a differentiable embedding in E^q, then it admits an isometric C^1 embedding in E^{q+1} [107], [122]. Combining this fact with the differentiable embeddability of smooth non-compact connected n-dimensional manifolds in E^{2n}, we see that any Riemannian manifold of dimension n admits an isometric C^1 embedding in E^{2n+1}. A C^1 orientable two-dimensional Riemannian manifold with finitely generated fundamental group admits a C^1 embedding in E^3.

An orientable (non-orientable) complete two-dimensional Riemannian manifold with finite fundamental group admits an isometric C^1 embedding in E^3 (E^4) with closed image [80].

1.2. Smooth surfaces of bounded extrinsic curvature. A qualitative abundance of immersions is ensured by lowering the smoothness or by increasing the codimension of an immersion. For many immersions of this kind one loses both the qualitative relationship of the structure of a surface with the sign of its curvature and the quantitative relationships between the intrinsic and extrinsic geometry of this surface. At the same time one can give examples in which isometric immersions are subjected to some additional 'regularity' conditions that ensure a rather close connection between the extrinsic and intrinsic geometry of an immersion. Examples of such restrictions are quite diverse, including both purely differential and geometric ones. Apparently, the first paper in which the 'regularity' of an immersion was understood purely geometrically (the convexity) was the paper of A. D. Aleksandrov devoted to Weyl's problem.

The smoothness of an immersion follows from the smoothness of the metric and from the convexity of the immersion. This is a classical result of Pogorelov.

Theorem 1.2.1 (Pogorelov [131], [133]). *If a convex surface has a regular metric of class C^k $(k \geqslant 2)$ with positive Gaussian curvature, then the surface itself is of class $C^{k-1,\beta}$ for any $\beta < 1$.*

We see that here the guaranteed smoothness of an immersion is lower that the smoothness of the immersed metric. However, it turns out that there is no loss of smoothness under an immersion.

Theorem 1.2.2 (Sabitov [142]). *A convex surface with $C^{k,\alpha}$ regular metric $(k \geqslant 2$, $0 < \alpha < 1)$ and of positive curvature is $C^{k,\alpha}$ smooth.*

A more general question naturally arises: What is the connection between the regularity of a metric induced by an immersion and the regularity class of this immersion? At first glance, the regularity of a metric must be reduced under an immersion. However, in fact this is not the case. This was shown by Sabitov and Shefel' in the paper [147], where, first of all, harmonic coordinates were introduced and used in the proof of the following assertion.

Theorem 1.2.3 (Sabitov and Shefel' [147]). *Every $C^{k,\alpha}$ smooth $(k \geqslant 2, 0 < \alpha < 1)$ l-dimensional surface F^l in a Riemannian space M^n whose regularity is not less than that of the surface is a $C^{k,\alpha}$ smooth isometric immersion of some $C^{k,\alpha}$ smooth Riemannian manifold M^l.*

Weaker results have been proved if the Gaussian curvature of a surface is only non-negative. The local immersibility as a C^{n-6} smooth and convex surface was proved for C^n smooth metrics with Gaussian curvature $K \geqslant 0$ and for $n \geqslant 10$ [110]. If a C^4 smooth metric with Gaussian curvature $K \geqslant 0$ is defined on a sphere, then there is a $C^{1,1}$ isometric embedding [89].

After the investigations of Nash and Kuiper it became clear that the condition of C^1 smoothness is too weak for an isometric immersion. In this case, the relationships between the intrinsic and extrinsic geometry are violated even if the codimension of an embedding is small. Any smooth surface F in E^3 admits a continuous bending in the class of isometric surfaces into a surface belonging to an arbitrarily small ball. In particular, a C^1 smooth surface with analytic metric of positive curvature can be non-convex.

It is quite clearly impossible to preserve the main integral theorem of surface theory connecting the intrinsic and extrinsic curvature in its usual formulation, even for smooth surfaces with arbitrarily nice intrinsic metric. This necessitates imposing other restrictions on the surface, involving its extrinsic geometry. The most natural and geometric condition is the assumption that the area of the spherical image is bounded, which enables one to readily and immediately define the most important notion of surface theory, namely, the notion of curvature [132], [133].

The exact definition of the class of surfaces under consideration is as follows. Let Φ be a smooth surface, and let $F_1, F_2, \dots, F_r$ be arbitrary pairwise disjoint closed sets on Φ. The spherical image $\overline{F}_s$ of any set F_s is also a closed set, and hence has a definite area $\mu(\overline{F}_s)$ (the Lebesgue measure). A surface Φ is referred to as a surface of *bounded extrinsic curvature* if

$$\mu(\overline{F}_1) + \mu(\overline{F}_2) + \dots + \mu(\overline{F}_r) < c(\Phi),$$

where $c(\Phi)$ is a constant that depends neither on the sets F_s nor on their number. Thus, this is a surface for which the area of the spherical image is finite with multiplicity taken into account.

For a surface of bounded extrinsic curvature one can introduce the notion of absolute extrinsic curvature on an arbitrary set. Namely, if G is an open set, $F_1, F_2, \dots, F_r$ are closed sets belonging to G, and $\mu(\overline{F}_1), \mu(\overline{F}_2), \dots, \mu(\overline{F}_r)$ are the areas of their spherical images, then by the absolute curvature of the surface on the set G one means the number

$$\sigma^0(G) = \sup \sum_s \mu(\overline{F}_s),$$

where the supremum is taken over all finite systems of closed disjoint sets in G. The absolute curvature on any set H is defined by

$$\sigma^0(H) = \inf_{H \subset G} \sigma^0(G),$$

where the infimum is taken over all open sets G containing H. Obviously, if a surface is regular (twice differentiable), then

$$\sigma^0(H) = \int_H |K| \, dS,$$

where K is the Gaussian curvature of the surface and dS is the area element.

The absolute curvature thus defined for surfaces has a property that is typical for curvatures, namely, it is completely additive on the ring of Borel sets.

The absolute curvature of a surface admits an integral representation (which is often useful) for any Borel set H:

$$\sigma^0(H) = \int_\omega n_H(X) \, dX,$$

where $n_H(X)$ is the number of points in H whose image under the spherical map is the point X of the unit sphere ω, dX is the area element of the sphere, and the integration is extended to the entire sphere.

The points of a surface of bounded extrinsic curvature are divided into regular and irregular points. A point X is said to be *regular* if in a sufficiently small neighbourhood of it there are no points at which the tangent planes are parallel to the tangent plane at X. The other points of the surface are said to be *irregular*. The absolute curvature of the surface on the set of irregular points is zero.

In turn, the regular points of a surface are divided into elliptic, hyperbolic, parabolic, and planar points, in dependence on the character of intersection of the tangent plane with the surface. Namely, if a point X of a surface has a neighbourhood $U(X)$ such that the tangent plane at X has a unique common point with $U(X)$, the point X itself, then this point is said to be *elliptic*. If the tangent plane at the point X intersects the surface near the tangent point along two simple curves originating from X, then the point X is said to be a *parabolic* point. If the intersection consists of four curves, then X is said to be *hyperbolic*. At any other regular point, the intersection of the tangent plane with the surface near the tangent point consists of an even number greater than four of simple curves originating from the tangent point. Any such point is said to be *planar*.

The above classification of regular points enables one to define in a natural way the notions of positive, negative, and total extrinsic curvature of a surface on an arbitrary set. Namely, by the positive (negative) extrinsic curvature of a surface on a set H one means the absolute curvature of the surface on the subset of elliptic (hyperbolic) points of H. The total extrinsic curvature on the set H is defined as the difference between the positive and negative curvatures of the surface on this set. Obviously, if the surface is regular, then the positive, negative, and total curvatures on a set H are equal to

$$\sigma^+(H) = \int_{H(K>0)} K\,dS, \quad \sigma^-(H) = -\int_{H(K<0)} K\,dS, \quad \sigma(H) = \int_H K\,dS,$$

respectively, where K is the Gaussian curvature of the surface, dS is the area element, and the integration in the first two formulae is taken over the subsets of H on which $K > 0$ and $K < 0$, respectively.

The positive, negative, and total curvatures of a surface of bounded extrinsic curvature are completely additive on the ring of Borel subsets of the surface.

The total curvature of a surface is the signed area of the spherical image; for a domain G homeomorphic to a disc and bounded by a simple curve γ with zero absolute curvature this area admits the representation

$$\sigma(G) = \int_{\omega-\overline{\gamma}} q_{\overline{\gamma}}(X)\,dX,$$

where $q_{\overline{\gamma}}(X)$ is the winding number of a point X of the unit sphere with respect to the curve $\overline{\gamma}$ that is the spherical image of the curve γ, which means that $q_{\overline{\gamma}}(X)$ is the number of times the curve $\overline{\gamma}$ passes anticlockwise around X, and dX is the area element of the sphere.

The extrinsic shape of a surface of non-negative curvature ($\sigma^- = 0$) and of zero curvature ($\sigma^- = \sigma^+ = 0$) is clarified by the following theorems.

Theorem 1.2.4 (Pogorelov [132]). *A surface of zero extrinsic curvature is developable, that is, it is locally isometric to a plane and has the usual structure of a regular developable surface with rectilinear generators and stationary tangent plane along any generator. A complete surface of zero curvature is cylindrical.*

Theorem 1.2.5 (Pogorelov [132]). *A complete surface with non-negative extrinsic curvature and positive curvature distinct from zero is either a closed convex surface or an infinite convex surface.*

Both the intrinsic geometric investigation of a surface of bounded extrinsic curvature and the determination of relationships between the intrinsic metric of a surface and its extrinsic shape are made possible by a theorem on the approximation of such a surface by regular ones. Namely: *let Φ be a surface of bounded extrinsic curvature and let X be an arbitrary point on Φ; then the point X has a neighbourhood $U(X)$ such that there is a sequence of regular, and even analytic, surfaces Φ_n with uniformly bounded absolute curvatures that converges to Φ in $U(X)$ together with the spherical images.*

This theorem enables one to conclude that any surface of bounded extrinsic curvature is a manifold of bounded intrinsic curvature in the A. D. Aleksandrov sense [3].

Thus, for a surface of bounded extrinsic curvature one has correctly defined notions of intrinsic positive curvature (ω^+), negative curvature (ω^-), and the total curvature (ω). For regular surfaces, the relationship between these curvatures and the corresponding extrinsic curvatures is established by the well-known Gauss theorem. Similar theorems hold for a surface of bounded extrinsic curvature.

For any Borel set H on a surface of bounded extrinsic curvature one has

$$\omega^+(H) = \sigma^+(H), \quad \omega^-(H) \geqslant \sigma^-(H), \quad \omega(H) \leqslant \sigma(H).$$

There is an open set G containing all regular points of the surface and such that for any Borel set H in G one has

$$\omega^+(H) = \sigma^+(H), \quad \omega^-(H) = \sigma^-(H), \quad \omega(H) = \sigma(H).$$

The intrinsic and extrinsic curvatures coincide for any Borel set on a closed surface of bounded extrinsic curvature.

We consider a surface with non-negative intrinsic curvature and bounded extrinsic curvature. Let us start from a surface of zero intrinsic curvature, that is, a surface for which both the positive and negative intrinsic curvatures vanish. Such a surface can have neither elliptic nor hyperbolic points. Indeed, if the surface has a hyperbolic point, then the negative extrinsic curvature of the surface is non-zero. Hence, the surface has an uncountable set of hyperbolic points, and the area of their spherical image is non-zero. On the set of hyperbolic points of the surface, the total intrinsic curvature and the total extrinsic curvature coincide. Since the extrinsic curvature is certainly negative, it follows that the intrinsic curvature is negative as well. However, this is impossible because the total curvature must vanish on any set of points of the surface. Thus, there can be no hyperbolic points on a surface with zero intrinsic curvature.

One can establish similarly that there can be no elliptic points on a surface of zero intrinsic curvature. Therefore, the positive and negative extrinsic curvatures of such a surface vanish, and the following assertions hold.

Theorem 1.2.6 (Pogorelov [132]). *A surface of bounded extrinsic curvature and zero intrinsic curvature is developable. This surface has the standard structure of a regular developable surface with rectilinear generators and stationary tangent planes along any generator.*

Theorem 1.2.7 (Pogorelov [132]). *A complete surface of class C^1 with zero intrinsic curvature and bounded extrinsic curvature is cylindrical.*

Let us now consider surfaces for which the positive intrinsic curvature is non-zero and the negative curvature vanishes. Such a surface can have no hyperbolic points. This can be proved by using the same arguments as those used above for a surface of zero curvature.

Thus, a surface of non-negative intrinsic curvature is also a surface of non-negative extrinsic curvature. Therefore, the corresponding theorems hold for these surfaces ([132], Chap. 2, § 4).

Theorem 1.2.8 (Pogorelov [132]). *A surface with non-negative* (*non-zero*) *intrinsic curvature and of bounded extrinsic curvature with boundary belonging to a plane is a convex surface*, *that is*, *it is a domain on the boundary of a convex body.*

Theorem 1.2.9 (Pogorelov [132]). *A complete surface with bounded extrinsic curvature and* (*non-zero*) *non-negative intrinsic curvature is either a closed convex surface or an infinite convex surface.*

1.3. Affine stable immersions of Riemannian metrics [40]. Another approach to the regularity of isometric immersions that guarantees a definite relationship between intrinsic and extrinsic geometric characteristics of submanifolds was suggested by Shefel' [152], [153], [155].

A property of a surface is said to be *geometric* if it is preserved under the transformations of E^n belonging to some group G. It is always assumed that G contains the group of homotheties and differs from the latter. Such groups are said to be *geometric*. A classification of geometric groups is obtained in [149], [150]. We note that the only meaningful cases to consider are groups of affine transformations, pseudo-groups of Möbius transformations (which are generated by the homotheties and inversions for $n > 2$), and, for somewhat other purposes, the groups of all diffeomorphisms of a chosen smoothness.

Definition. A surface F in E^n is said to be a G-*stable* immersion of a metric of some class $\mathcal{H}$ if any transformation belonging to the group G takes F to a surface whose intrinsic metric also belongs to the class $\mathcal{H}$.

Here it is tacitly assumed that G is a geometric group (or a pseudo-group) of transformations of E^n. Since the identity transformation id belongs to G, it obviously follows that the intrinsic metric of the surface F itself belongs to $\mathcal{H}$. In this definition, the class $\mathcal{H}$ of metrics is not necessarily exhausted by Riemannian metrics. Correspondingly, by an *immersion* of a metric one means here a C^0 smooth (topological) immersion that is an isometry.

It is essential that the condition of G-stability of a surface imposes no *a priori* restrictions on the dimension n of the enveloping space. We note that G-stable immersions of the metrics of some class $\mathcal{H}$ (that differs from the class of all admissible metrics) always have some general geometric property. The passage from arbitrary immersions to G-stable immersions enables one to establish the dual relationship between extrinsic and intrinsic properties of the surface.

The fact that the notion of G-stability is natural is illustrated by the following assertions established by Shefel' in [147], [151], [152] in a more general form.

Theorem 1.3.1 (Shefel' [151]). *The only affine stable immersions in E^n, $n \geqslant 3$, for the class of two-dimensional Riemannian metrics of positive curvature are locally convex surfaces in some $E^3 \subset E^n$.*

On the contrary, the class of affine stable immersions for two-dimensional Riemannian metrics of negative curvature is far from being exhausted by surfaces in E^3; however, all such immersions belong to the class of the so-called saddle surfaces, that is, surfaces that locally admit no strict support hyperplanes.

Theorem 1.3.2 (Shefel' [152]). *Let G be a group of C^∞ diffeomorphisms of E^n. Then any G-stable immersion for the class of Riemannian metrics of class $C^{l,\alpha}$, $l \geqslant 2$, $0 < \alpha < 1$, is a surface of the same smoothness.*

The most interesting situation is the case in which the class $\mathcal{H}$ of metrics, the group G, and the class $\mathcal{M}$ of surfaces satisfy the following conditions.

1. $\mathcal{M}$ coincides with the class of G-stable immersions of the metrics in the corresponding class $\mathcal{H}$ of metrics.

2. Each metric of class $\mathcal{H}$ admits an immersion in the form of a surface of class $\mathcal{M}$.

In this case the class $\mathcal{M}$ of surfaces and the class $\mathcal{H}$ of metrics are said to be *G-connected*.

1.4. Classes of surfaces and classes of metrics. The classification of points enables one to distinguish six classes of smooth surfaces. The surfaces of the first three classes M^+, M^-, and M^0 consist only of elliptic, hyperbolic, and parabolic points, respectively. The surfaces of class M_0^+ consist of elliptic and parabolic points only, and the surfaces of class M_0^- of hyperbolic and parabolic points only. Finally, the class M is formed by all smooth surfaces.

The surfaces of class M_0^+ are said to be *normal* surfaces of non-negative curvature, and the surfaces of classes M_0^- and M^- are said to be *saddle* and *strictly saddle* surfaces, respectively.

Theorem 1.4.1 (Shefel' [152]). *The class M^+ in E^n consists of locally convex surfaces belonging to some $E^3 \subset E^n$. A complete surface of class M^+ is a complete convex surface (the boundary of a convex body in E^3). Normal surfaces of non-negative curvature (of class M_0^+) are characterized by the condition that either any point of such a surface has a neighbourhood in the form of a convex surface or there is a rectilinear generator passing through this point and with ends at the boundary of the surface, and the tangent plane along the generator is stationary. A complete surface of class M_0^+ is either a convex surface in E^3 or a cylinder in E^n. The class M_0 consists of developable surfaces. The complete surfaces of this class are cylinders.*

The saddle surfaces F (the class M_0^-) can be characterized by the condition that no hyperplane cuts off a *heel* of F, that is, a domain whose closure is compact and disjoint from the boundary of F.

According to the sign of the Gaussian curvature one can distinguish the following natural classes of two-dimensional Riemannian metrics: the classes K^+, K^-, and K^0 of Riemannian metrics of positive, negative, and zero curvature; the classes K_0^+ and K_0^- of metrics of non-negative and non-positive curvature; and the class K of all Riemannian metrics. The classes of surfaces and metrics labelled by the same indices are said to be *associated*.

The local properties of smooth surfaces and metrics are usually reduced to conditions on a surface (or a metric) at any point. As a rule, these conditions describe the behaviour of a surface (a metric) in a neighbourhood of any point up to second-order terms. In what follows, a geometric property of a surface is said to be *local* if it is a property of a point of the surface and the validity of this property at some point p of a surface F implies its validity at p for any other surface that coincides with F in a neighbourhood of p up to infinitesimals of second order.

For classes of surfaces and metrics singled out on the basis of local properties we distinguish between G-relationship in the small and G-relationship in the large, and we pose two problems, in the small and in the large, respectively. A class $\mathcal{M}$ of surfaces and a class $\mathcal{H}$ of metrics are said to be *G-connected in the small* if: 1) the class $\mathcal{M}$ of surfaces coincides with the class of G-stable immersions of metrics of class $\mathcal{H}$; 2) every metric of class $\mathcal{H}$ admits a local immersion as a surface of class $\mathcal{M}$. The problem in the small is to find classes of surfaces and metrics that are G-connected in the small.

A class $\widetilde{\mathcal{M}}$ of complete surfaces and a class $\widetilde{\mathcal{H}}$ of complete metrics are said to be *G-connected in the large* if 1) $\widetilde{\mathcal{M}}$ coincides with the class of G-stable immersions of metrics of class $\widetilde{\mathcal{H}}$, and 2) every metric of class $\widetilde{\mathcal{H}}$ admits an immersion (in the large) as a surface of class $\widetilde{\mathcal{M}}$. The problem in the large is to find classes of surfaces and metrics that are G-connected in the large. In contrast to the problem in the small, one needs as a rule to impose *a priori* conditions of non-local character on classes of complete surfaces and metrics that are G-connected in the large even for cases in which local properties are the basis for distinguishing these classes.

As in the general case, the main problem on the correspondence between surfaces and metrics for smooth surfaces and for the affine group of transformations consists of problems in the small and in the large. The problem in the small is completely solved for the classes K^+, K^-, K^0, and K; the following two theorems hold.

Theorem 1.4.2 [40]. *The classes M^+, M^-, M^0, and M of smooth surfaces and the corresponding classes of metrics are pairwise affinely associated in the small.*

Theorem 1.4.3 [40]. *If one restricts oneself to classes of smooth surfaces each defined by a local geometric property, then there are no pairs affinely associated in the small that are distinct from the pairs listed in Theorem* 1.4.2, *except possibly for the pair K_0^-, M_0^-.*

Theorem 1.4.2 combines the following assertions.

1. Each of the above classes of surfaces is affinely invariant.

2. The intrinsic metric of any surface in any of these classes belongs to the associated class of metrics.

3. An affine stable immersion in E^n of a metric of any of these classes belongs to the associated class of surfaces.

4. Any metric of any of the above classes admits a local immersion as a surface of the associated class in E^n.

Theorem 1.4.4 [40]. *The classes $\widetilde{M}^+$, $\widetilde{M}_0$, and $\widetilde{M}$ of smooth complete simply connected surfaces and the associated classes of Riemannian metrics are affinely connected in the large.*

A C^1 surface in E^3 is said to be a *normal* surface of non-negative curvature if for any point x of the surface at least one of the following assertions holds: a) x has a neighbourhood in the form of a convex surface (in particular, this neighbourhood can be a plane domain); b) x belongs to a rectilinear generator along which the tangent plane is stationary. By a *normal developable surface* one means a normal surface of non-negative curvature that has no points of convexity, and hence is a saddle surface. As is well known, a C^2 surface is a normal surface of non-negative curvature (a normal developable surface) if and only if its Gaussian curvature is non-negative (vanishes, respectively).

One can readily see that, if a point of a normal surface of non-negative curvature has no neighbourhood in the form of a convex surface, then the rectilinear generator passing through this point is unique, and no other point of this rectilinear generator has a neighbourhood of this kind; this rectilinear generator can be extended to the boundary of the surface. A normal surface of non-negative curvature is a manifold of non-negative curvature with respect to its intrinsic geometry, and a normal developable surface is locally isometric to a plane. A complete normal surface of non-negative curvature and of non-zero intrinsic curvature is a convex surface, and a complete normal developable surface is a cylinder. Each heel that is cut off from a normal surface of non-negative curvature by a plane is a convex surface.

We recall that K_0^+, K_0^-, and K^0 stand for the classes of two-dimensional manifolds of non-negative, non-positive, and zero curvature, respectively.

Theorem 1.4.5 (Shefel' [153]). *Let a C^1 surface F in E^3 have finite extrinsic positive curvature μ^+. If the intrinsic metric of F belongs to one of the classes K_0^+, K_0^-, and K^0, then F is a normal surface of non-negative curvature, a saddle surface, or a normal developable surface, respectively.*

Theorem 1.4.6 (Shefel' [153], [154]). *If a C^1 surface F in E^3 has finite extrinsic positive curvature and belongs to one of the classes K_0^+, K_0^-, and K^0 with respect to its intrinsic geometry, then the image of F under an affine transformation has an intrinsic metric of the same class.*

In other words, the theorem states that C^1 surfaces of bounded positive extrinsic curvature are affine stable immersions for the metrics in the classes listed in the theorem.

Theorem 1.4.7 (Shefel' [153], [154]). *If a C^1 surface F is an affine stable immersion in E^3 of a metric of one of the classes K_0^+, K_0^-, and K, then F is a normal surface of non-negative curvature, a saddle surface, and a normal developable surface, respectively.*

This theorem has the following corollary.

Theorem 1.4.8 (Shefel' [153], [154]). *A C^1 isometric affine stable immersion of a complete two-dimensional metric of zero curvature in E^3 is a cylinder.*

A similar result holds if one replaces the enveloping space E^3 by E^n $(n \geqslant 3)$.

A deep exposition of the problems of this chapter can be found in the survey of Burago [40], the initial version of which was written together with Shefel'.

CHAPTER 2

ISOMETRIC REGULAR IMMERSIONS OF TWO-DIMENSIONAL METRICS OF CONSTANT CURVATURE

2.1. Classical results. Let us first consider isometric immersions in the class of regular surfaces of class C^r $(r \geqslant 2)$. The following assertion holds.

Theorem 2.1.1. *A complete connected regular surface of constant Gaussian curvature $K > 0$ in the Euclidean space E^3 is a standard sphere.*

This theorem was first proved by Minkowski. Liebmann gave a proof in the analytic case, and then Hilbert suggested his now classical proof.

Instead of regularity of the surface one can suppose only local convexity of this surface or assume that it is smooth of class C^1 and of bounded extrinsic curvature or that it is a C^1 smooth affine stable immersion in the class of metrics of positive curvature. In the first case it follows from Theorem 1.2.9 that the surface is closed and convex. Furthermore, it follows from the Pogorelov theorem on unique determination of general closed convex surfaces [133] that the surface is a standard sphere. In the other case, it follows from Theorem 1.4.7 that the surface is convex, and then we again apply the theorem on the unique determination. The assertion of the theorem now follows from the results of Pogorelov [131]–[133] and Shefel' [153], [154] formulated in the previous chapter.

Theorem 2.1.2. *A complete connected regular surface of zero Gaussian curvature in the Euclidean space E^3 is a cylinder.*

The regularity condition can be weakened by assuming only that the surface belongs to the class of smooth surfaces of bounded extrinsic curvature [132]. The validity of this theorem in the class of smooth surfaces of bounded extrinsic curvature is asserted by Theorem 1.2.7, and in the class of C^1 affine stable immersions of metrics of zero curvature by Theorem 1.4.8. One can further weaken this regularity condition and replace it by the condition that a line of the enveloping space belongs to the surface. The following assertion holds.

Lemma 2.1.1 [29]. *Let F be a two-dimensional surface of class C^0 in E^n that is isometric to a plane and contains a line p of the enveloping space. Then F is a cylinder whose generator is parallel to the line p.*

Proof. Let $\overline{F}$ be a plane isometric to the surface F, let $\overline{Q} \in \overline{F}$ be the point corresponding to $Q \in F$ under this isometry, let $\overline{P}\,\overline{Q}$ be the distance on $\overline{F}$ between points $\overline{P}$ and $\overline{Q}$, and let PQ be the distance between points P and Q in E^n. We introduce the arclength parameter s on $\overline{p}$ measured from a chosen point $\overline{P}(0) \in \overline{p}$. Let $M(s)$ be the shortest path perpendicular to $\overline{p}$ that passes through the point $\overline{P}(s) \in \overline{p}$. We pass a hyperplane $E^{n-1}(s)$ orthogonal to p through any point $P(s) \in p$ of the surface F, and let $F(s) = E^{n-1}(s) \cap F$. We claim that $F(s) = M(s)$ for any s. Suppose that for some s_0 in $F(s_0)$ one has $Q \in M(s_1)$, where $s_1 \neq s_0$. Let us take a triangle $\overline{P}(s_0)\,\overline{Q}\,\overline{P}(s)$ on $\overline{F}$. Since $\overline{Q} \notin \overline{M}(s_0)$, it follows that the shortest path $\overline{P}(s_0)\,\overline{Q}$ is not perpendicular to $\overline{p}$. We choose a point $\overline{P}(s)$ on a ray that forms an acute angle with $\overline{P}(s_0)\,\overline{Q}$.

If s is sufficiently large, then the angle $\overline{P}(s_0)\overline{Q}\,\overline{P}(s)$ becomes obtuse, and

$$\overline{P}(s_0)\overline{P}(s) > \overline{Q}\,\overline{P}(s).$$

Let us consider the triangle $P(s_0)QP(s)$. It is a right triangle with right angle $QP(s_0)P(s)$. Thus,

$$P(s)Q > P(s_0)P(s);$$

however,

$$P(s_0)P(s) = \overline{P}(s_0)\overline{P}(s), \qquad QP(s) \leqslant \overline{Q}\,\overline{P}(s).$$

Hence, $\overline{P}(s)\overline{Q} > \overline{P}(s_0)\overline{P}(s)$. We arrive at a contradiction, that is,

$$F(s) = M(s).$$

Let $\overline{Q}$ be an arbitrary point of $\overline{F}$, and let $\overline{q}$ be a line passing through $\overline{Q}$ and parallel to $\overline{p}$. We claim that q is a line (in the Euclidean space) parallel to p. Let $\overline{Q}_1$ and $\overline{Q}_2$ be arbitrary points on $\overline{q}$. If one drops the perpendiculars from $\overline{Q}_1$ and $\overline{Q}_2$ to $\overline{p}$, and if $\overline{P}_1$ and $\overline{P}_2$ are the feet of the perpendiculars, then $\overline{P}_1\overline{P}_2 = \overline{Q}_1\overline{Q}_2$.

Let us consider the spatial quadrangle $P_1P_2Q_2Q_1$. Let the Euclidean space E^3 be the span of this quadrangle. For the origin of the rectangular system of coordinates in E^3 we take P_1 and direct the axis z along the line p. The points P_1, P_2, Q_1, and Q_2 have the following coordinates:

$$P_1(0,0,0), \quad P_2(0,0,z_2), \quad Q_1(x_1,y_1,0), \quad Q_2(x_2,y_2,z_2),$$
$$P_1P_2 = |z_2|, \quad Q_1Q_2 = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+z_2^2} \geqslant P_1P_2;$$

however, $P_1P_2 = \overline{P}_1\overline{P}_2$ and $Q_1Q_2 \leqslant \overline{Q}_1\overline{Q}_2$, hence $Q_1Q_2 \leqslant P_1P_2$. Therefore, $Q_1Q_2 = P_1P_2$. This is possible only if $x_1 = x_2$ and $y_1 = y_2$, that is, the segment Q_1Q_2 is parallel to p, and it is a shortest path on $\overline{F}$.

Since the points $\overline{Q}_1$, $\overline{Q}_2$, and $\overline{Q}$ are arbitrary, this proves the lemma [29], [36].

Minding investigated some properties of surfaces of revolution with Gaussian curvature $K = \text{const} < 0$ already at the end of the 1930s. One of these surfaces (the pseudo-sphere) was studied by Beltrami in more detail. They contain singular curves and points at which the regularity of the surface is violated. Hilbert was the first to pose the problem of whether the Lobachevsky plane can be realized in the Euclidean space E^3 as a surface without singularities, and he answered this question in the negative.

Theorem 2.1.3 (Hilbert [88]). *In the Euclidean space E^3 there is no analytic complete surface of constant negative curvature.*

Additional investigations [105], [83] show that the following assertion holds.

Theorem 2.1.4 [141]. *In E^3 there is no C^2 surface isometric in the large to the Lobachevsky plane.*

If one restricts oneself to the assumption that the surface under consideration is only of class C^1, then it follows from results of Kuiper that an isometric immersion of the Lobachevsky plane in E^3 is possible in this class of surfaces. It is not known whether Theorem 2.1.4 can be generalized to smooth surfaces of class C^1 that are of local bounded extrinsic curvature [141].

Let M^3 be a complete simply connected Riemannian manifold of constant sectional curvature K_0. For a regular surface $F \subset M^3$ one can define the intrinsic curvature (the Gaussian curvature) K_{int} and the extrinsic curvature K_{ext} as the product of the principal curvatures. It follows from the Gauss equation that

$$K_{\text{int}} = K_{\text{ext}} + K_0.$$

If $K_0 = +1$, then M^3 is the standard sphere S^3; if $K_0 = -1$, then M^3 is the hyperbolic space H^3.

The following assertions hold for spaces of constant curvature.

Theorem 2.1.5 [159]. *Any complete regular surface of class C^2 in S^3 or H^3 of constant Gaussian curvature $K_{\text{int}} > 0$ and constant $K_{\text{ext}} \geqslant 0$ is an umbilical surface.*

Theorem 2.1.6 [159]. *There is no complete regular surface of class C^2 in S^3 or H^3 with constant $K_{\text{int}} < 0$ and $K_{\text{ext}} < 0$.*

For the spherical space S^3 we immediately obtain the following assertion: if $K_{\text{ext}} > 0$, then the surface is a small sphere; if $K_{\text{ext}} = 0$, then the surface is a totally geodesic great sphere S^3; if $-1 < K_{\text{ext}} < 0$, then there is no complete surface in S^3. Indeed, in this case $K_{\text{int}} > 0$, and the universal covering of the surface is homeomorphic to a sphere. On the other hand, one has $K_{\text{ext}} < 0$, and therefore the principal curvatures are of opposite sign. We thus have a continuous direction field on the sphere S^2, which is impossible because the Euler characteristic of the sphere is non-zero. If $K_{\text{ext}} < -1$, then it follows from Theorem 2.1.6 that there is no such surface.

Figure 1. Domain with minimality condition

2.2. Flat metrics in the sphere. It remains to study the case in which $K_{\text{ext}} = -1$ and $K_{\text{int}} = 0$. There are many complete flat regular surfaces in the spherical space S^3. It follows from the classical Beltrami–Enneper theorem that the torsion of asymptotic curves on a surface of constant curvature is constant and has opposite sign for distinct families. The analytic curves of the same family are obtained from one another by a Clifford shift, that is, by an isometry $A\colon S^3 \to S^3$ such that the distance $d(x, A(x))$ on the sphere is constant [159], [160].

Let us describe the flat tori in S^3. We consider the sphere S^2 as the set of unit imaginary quaternions and the sphere S^3 as the entire set of unit quaternions. In this case, the Hopf bundle

$$h\colon S^3 \to S^2$$

becomes

$$h(q) = \overline{q}\, i\, q, \qquad q \in S^3,$$

where i is the imaginary unit, $i^2 = -1$.

We regard S^1 as the set of unit complex numbers and $c\colon S^1 \to S^2$ as an immersion of class C^2. Since $S^2 \subset E^3$, we can define the indicatrix of the tangents $c^1\colon S^1 \to S^2$. Let us consider the Hopf torus $T = h^{-1}[c(S^1)]$, which is a special case of a flat torus in S^3, possibly with self-intersections [130]. On T one has two families of asymptotic curves, one of which is the family of great circles and the other is a family of asymptotic curves that are also closed.

Let $\gamma\colon S^1 \to S^3$ be a regular parametrization of any of these asymptotic curves. It is clear that $h\colon \gamma(S^1) \to c(S^1)$ is a covering. We give another description of the asymptotic curves γ with respect to c. Let US^2 be the bundle of unit vectors over S^2, and let $SO(3)$ be the group of orthogonal transformations of E^3 with determinant $+1$. For the map $c\colon S^1 \to S^2$ we define $\widehat{c}\colon S^1 \to SO(3)$ by the formula $\widehat{c} = (c, c', cc')$. Let $\pi\colon S^3 \to SO(3)$ be

$$\pi(q) = (\widetilde{q}\, i\, q, \widetilde{q}\, j\, q, \widetilde{q}\, k\, q).$$

It is clear that π is a double covering and that $\pi(q) = \pi(-q)$ for any q. If $\rho_\theta\colon US^2 \to US^n$ is a rotation through an angle θ of any fiber US^2, then for any γ there is a θ such that

$$\pi(\gamma(S^1)) = \rho_\theta(\widehat{c}(S^1)).$$

We recall that any regular curve $c\colon S^1 \to S^2$ of class C^1 is regularly homotopy equivalent to a circle under one circuit ($c \sim 0$) or under m cicruits ($c \sim m-1$).

If $c \sim 0$, then $\pi\colon \gamma(S^1) \to \rho_\theta \circ \widehat{c}(S^1)$ is a double covering. If $c \sim 1$, then $\pi\colon \gamma(S^1) \to \rho_\theta \circ \widehat{c}(S^1)$ is a diffeomorphism.

Let us choose C^2 immersions $a, b\colon S^1 \to S^2$. Let $\alpha, \beta\colon S^1 \to S^3$ be closed asymptotic curves of the Hopf tori. We define

$$\overline{\alpha} \cdot \beta\colon S^1 \times S^1 \to S^3$$

by the formula

$$\overline{\alpha} \cdot \beta(s,t) = \overline{\alpha}(s) \cdot \beta(t), \qquad (s,t) \in S^1 \times S^1.$$

Let $k_c\colon S^1 \to \mathbb{R}$ be the geodesic curvature of the curve $c\colon S^1 \to S^2$.

Theorem 2.2.1 [159]. *Let $a, b\colon S^1 \to S^2$ be C^2 immersions. If $k_a(S^1) \cap k_b(S^1) = \varnothing$, then $\overline{\alpha} \cdot \beta$ is an immersion with flat metric. Up to a motion in S^3, all immersed flat tori can be obtained in this way.*

One can show that, if $a \sim 0$ and $b \sim 0$, then $\overline{\alpha} \cdot \beta$ is a double covering of the image, and the image admits a central symmetry [172].

Weiner completely determined the immersed flat tori in S^3 by their Grassmann image. Let α be a regular closed curve on the standard sphere S^2. Let ds and k be the arclength element and the geodesic curvature of the curve α, respectively. We set

$$\mu(\alpha) = \max\left\{ \int_\beta k\, ds : \beta \text{ an arc of the curve } \alpha \right\}, \qquad \tau(\alpha) = \int_\alpha k\, ds.$$

If γ is the indicatrix of the tangents of a regular closed curve in S^2, then $\mu(\gamma) < \pi$. Let $X\colon T \to S^3$ be a flat torus in S^3. Regarding X as a surface in E^4, we obtain the Grassmann image M of T in the Grassmann manifold $G(2,4)$,

$$G\colon T \to G(2,4),$$

where $G(2,n) = S_1 \times S_2$ is the metric product of two standard spheres of radius $\frac{1}{\sqrt{2}}$. We say that a surface in $G(2,n)$ satisfies the condition E if $M = \gamma_1 \times \gamma_2$, where the γ_i are closed curves in S_i $(i = 1, 2)$ for which $\tau(\gamma_i) = 0$ and $\mu(\gamma_1) + \mu(\gamma_2) < \pi$.

Let $[\gamma_i]$ be the homotopy class of the parametrization γ_i in $\pi_1(\gamma_1 \times \gamma_2)$. Let H_{11} be the subgroup of $\pi_1(\gamma_1 \times \gamma_2)$ generated by $\{[\gamma_1]+[\gamma_2], [\gamma_1]-[\gamma_2]\}$, let H_{12} be the subgroup generated by $\{2[\gamma_1], [\gamma_2]\}$, let $H_{22} = \pi_1[\gamma_1 \times \gamma_2]$, and let $H_{mn} = \{m[\gamma_1], n[\gamma_2]\}$.

Theorem 2.2.2 (Weiner [172]). *Let $G\colon T \to G(2,4)$ be a smooth map of the torus into the Grassmann manifold $G(2,4)$. Then G is a Grassmann image of a flat torus $X\colon T \to S^3 \subset E^4$ if and only if $G(T)$ satisfies the condition* E, *$G \to \gamma_1 \times \gamma_2$ is a covering, and $G_*(\pi_1(T)) \subset H_{mn}$ if $\gamma_1 \sim m$ and $\gamma_2 \sim n$.*

A sufficient condition in the case when $\mu(\gamma_1) < \frac{\pi}{2}$ and $\mu(\gamma_2) < \frac{\pi}{2}$ was proved in [65]. In the class of flat tori, the Hopf tori are distinguished by the condition that the curve γ_1 in the Grassmann image $\gamma_1 \times \gamma_2$ is a great circle. It follows from Theorem 2.2.2 that there is no flat Klein bottle in the standard projective space P^3 with a metric of constant curvature [172].

Definition. 1. By a *Clifford torus* one means a Riemannian manifold isometric to E^2/Γ, where Γ is the orthogonal lattice of diameter π.

2. By a *product of circles* in S^3 one means the direct product of two circles in E^4 centred at the origin and located in orthogonal planes. It is assumed that if r_1 and r_2 are the radii of these circles, then $r_1^2 + r_2^2 = 1$.

It is clear that a product of circles is a Clifford torus. In a sense, the converse is also true.

Theorem 2.2.3 (Weiner [173]). *Let $X\colon T \to S^2$ be an isometric immersion of a flat torus of diameter $\leqslant \pi$. If $X(T)$ contains a pair of antipodal points, then $X(T)$ is a product of circles.*

Theorem 2.2.3 implies the following.

Corollary 2.2.1 [173]. *An embedded Clifford torus is a product of circles.*

Corollary 2.2.2. *There are no other embedded flat tori of diameter $\leqslant \pi$ in S^3 except for Clifford tori.*[1]

In [104] extrinsic geometric properties of flat tori were studied, and the following assertion was proved.

Theorem 2.2.4 (Kitagawa [104]). *If $f\colon M \to S^3$ is an isometric embedding of a flat torus M in the spherical space S^3, then the image $f(M)$ is invariant with respect to the central symmetry in S^3.*

In the theorem one cannot replace an embedded surface by an immersed one, because there is an example of an immersed flat torus not invariant under the central symmetry. However, the following question remains open: Is it true that for any immersed flat torus there is a pair of antipodal points of S^3 that belong to the surface [104]?

For $K_{\text{ext}} = 0$ there is an analogue of Lemma 2.1.1.

Theorem 2.2.5 ([30], [36]). *Let F be a C^1 surface in $S^3(1)$ isometric to the sphere $S^2(1)$. If F contains a great circle of the enveloping sphere S^3, then F is a totally geodesic great sphere.*

Proof. Let S^1 be a great circle in S^3 that belongs to F, let $O_1, O_2 \in F$ be the points polar to S^1, let $\rho(x,y)$ be the distance on the surface F, and let r be the distance in S^3. We take two antipodal points X and Y on S^1. Then

$$\rho(X,O_1) = \frac{\pi}{2}\,, \quad \rho(Y,O_1) = \frac{\pi}{2}\,; \qquad r(X,O_1) = r_1, \quad r(Y,O_1) = r_2;$$
$$r_1 \leqslant \frac{\pi}{2}\,, \quad r_2 \leqslant \frac{\pi}{2}\,.$$

But $r_1 + r_2 = \pi$, and hence $r_1 = \frac{\pi}{2}$ and $r_2 = \frac{\pi}{2}$. Let us draw a sphere $S^2(O_1)$ containing S^1 and O_1. All shortest paths XO_1, $X \in S^1$, on the sphere $S^2(O_1)$ belong to the surface F, and hence the entire hemisphere of $S^2(O_1)$ that contains O_1 belongs to F. A part of the surface F that contains O_2 is also a hemisphere. However, since F is of class C^1, it follows that $S^2(O_1) = S^2(O_2)$, and F is a totally geodesic great sphere.

For the hyperbolic space H^3 one has

$$K_{\text{int}} = K_{\text{ext}} - 1.$$

If $K_{\text{ext}} > 1$, then a complete regular surface in H^3 is a sphere. If $K_{\text{ext}} < 0$, then there is no such surface. It remains to study the interval $0 \leqslant K_{\text{ext}} \leqslant 1$. For $K_{\text{ext}} = 1$ and $K_{\text{int}} = 0$, a complete regular surface in H^3 is either a horosphere or an equidistant curve of a geodesic (Volkov and Vladimirova, [170]). There are many complete surfaces in H^3 with $K_{\text{ext}} = 0$, which can be of arbitrarily high genus.

Isometric immersions of the Lobachevsky plane $H^2(-1)$ into $H^3(-1)$ were studied by Nomizu [125]. These surfaces are 1-strongly parabolic. The curvature of a

[1] *Translator's note*: of diameter π. See K. Enomoto, Y. Kitagawa, and J. L. Weiner, Proc. Amer. Math. Soc. **124** (1996), no. 1, pp. 265–268.

curve orthogonal to complete totally geodesic leaves does not exceed 1, where the curvature is taken with respect to the metric of the surface. In [125] a one-parameter family of isometric immersions of $H^2(-1)$ into $H^3(-1)$ is explicitly given. In [92] the description of isometric immersions of $H^2(-1)$ into $H^{-3}(-1)$ was transformed into the solution of the degenerate Monge–Ampère equation in the unit disc. Corresponding to any regular isometric immersion of $H^2(-1)$ into $H^3(-1)$ is a solution of the Monge–Ampère equation

$$\left(\frac{\partial^2 u}{\partial \xi_i \, \partial \xi_j}\right) = 0, \qquad \xi = (\xi_1, \xi_2) \in D. \tag{2.2.1}$$

Conversely, corresponding to any regular solution U of the equation (2.2.1) is a regular isometric immersion of $H^2(-1)$ into $H^3(-1)$ with the first fundamental form

$$g_{ij} = \lambda^{-1}(\lambda^2 \delta_{ij} \xi_i \xi_j), \qquad \lambda = \sqrt{1 - \xi_i^2 - \xi_2^2},$$

and the second fundamental form (see [92])

$$h_{ij} = \lambda^{-1} \frac{\partial^2 u}{\partial \xi_i \, \partial \xi_j}, \qquad i, j = 1, 2.$$

The analogous problem was also solved for isometric immersions of $H^2(-1)$ into $H^3(c)$ [91]. Corresponding to any isometric immersion of $H^2(-1)$ into $H^3(c)$ is a solution of the Monge–Ampère equation

$$\det\left(\frac{\partial^2 u}{\partial \zeta_i \, \partial \zeta_j}\right) = \frac{-c-1}{(1 - \zeta_1^2 - \zeta_2^2)^2} \tag{2.2.2}$$

in some domain D.

Conversely, corresponding to any smooth solution of (2.2.2) is a smooth isometric immersion of $H^2(-1)$ into $H^3(c)$ with the following first and second fundamental forms g and h:

$$\begin{aligned} g_{ij} &= \lambda^{-4}(\lambda^2 \delta_{ij} + \zeta_i \zeta_j), \quad \lambda = \sqrt{1 - \zeta_1^2 - \zeta_2^2}, \\ h_{ij} &= \lambda^{-1} \frac{\partial^2 u}{\partial \zeta_i \, \partial \zeta_j}, \quad i, j = 1, 2. \end{aligned} \tag{2.2.3}$$

For $0 < K_{\text{ext}} < 1$, equidistant loci and surfaces of revolution are known. It is not known whether or not other complete surfaces exist.

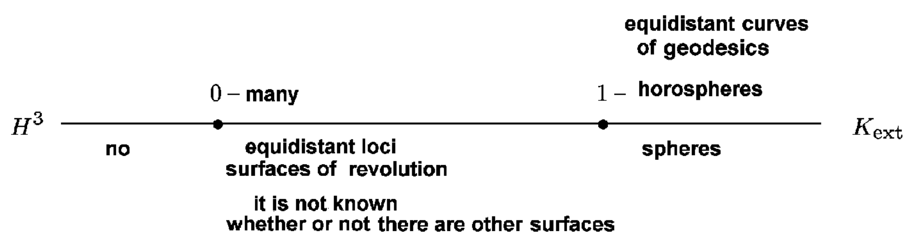

Figure 2. Domain with the minimal property

2.3. Isometric immersions of two-dimensional flat metrics. Subtle results on immersions of locally Euclidean metrics were obtained by Sabitov [144].

Theorem 2.3.1 (Sabitov [144]). *If a domain with locally Euclidean metric admits an isometric immersion into E^2, then it can be isometrically embedded in E^3. Any simply connected domain with locally Euclidean metric can be isometrically embedded in E^3.*

Suppose that in a plane domain D (simply connected or multiply connected) we are given a locally Euclidean metric

$$ds^2 = E(u,v)\,du^2 + 2F(u,v)\,du\,dv + G(u,v)\,dv^2, \qquad (u,v) \in D. \tag{2.3.1}$$

The condition that the metric (2.3.1) be locally Euclidean means that any point in D has a neighbourhood isometric to a disc on the plane E^2 with the standard Euclidean metric $dx^2 + dy^2$. Thus, local isometric embeddability of a locally Euclidean metric (2.3.1) into the Euclidean plane E^2 follows from the very definition of locally Euclidean metric. We are interested in the problem of isometric embeddings in E^2 and immersions into E^2 'in the large' for the entire domain D, or, in other words, the existence problem for a map

$$x = x(u,v), \qquad y = y(u,v) \tag{2.3.2}$$

that reduces (2.3.1) in the large to the form $dx^2 + dy^2$. For this map the formulae (2.3.2) give an isometric immersion of D into E^2 if the map is single-valued, and an embedding in E^2 if the map is one-sheeted in the entire domain D.

Let a locally Euclidean metric ds^2 be given in a circular n-connected domain D_n (that is, in a disc for $n = 1$ and in a domain bounded by n circles for $n \geqslant 2$) in the plane of the variables (ξ, η). Without loss of generality one can assume that the coordinates (ξ, η) are isothermal (the general case can be reduced to this special case by a quasi-conformal transformation). Then

$$ds^2 = \Lambda^2(\xi,\eta)(d\xi^2 + d\eta^2), \qquad \Lambda > 0, \quad \Delta \ln \Lambda = 0. \tag{2.3.3}$$

Furthermore, we assume that $\Lambda(\xi,\eta)$ is regular in the closed domain $\overline{D}_n$ in such a way that the unique map (up to a motion in E^2) of the form $z = x + iy = \Phi(\zeta)$, $\zeta = \xi + i\eta$, that reduces (2.3.3) to the standard form $ds^2 = dx^2 + dy^2$ is smooth of class C^m in $\overline{D}_n$, $m \geqslant 1$. We recall that the map $\Phi(\zeta)$ is determined via $\Lambda(\eta)$ by the relation $|\Phi'(\eta)| = \Lambda(\eta)$, and hence if $\Lambda \in C^{m,\alpha}$, $m \geqslant 0$, $0 < \alpha < 1$, then $\Phi(\eta) \in C^{m+1,\alpha}$.

Any condition that the holomorphic function $z = \Phi(\zeta)$ be single-valued gives an isometric immersion of the metric (2.3.3) into E^2, and any condition that this function be one-sheeted gives an embedding of (2.3.3) in E^2, and conversely. The following assertion holds.

Theorem 2.3.2 (Sabitov [144]). *A smooth locally Euclidean metric* (2.3.1) *of class $C^{m,\alpha}$, where $m > 0$ and $0 < \alpha \leqslant 1$, defined in a simply connected domain D with smooth boundary of class $C^{m+1,\alpha}$ can be isometrically immersed into E^2 by a smooth map of class $C^{m+1,\alpha}$.*

Let us consider in more detail the problem of embedding a metric (2.3.3) in E^2. We assume that the metric (2.3.3) is defined in the disc $\overline{\Omega}$ given by $\xi^2 + \eta^2 \leqslant 1$.

Let the rotation of any arc of the boundary of the disc D with respect to the metric (2.3.3) be non-negative. In this case D can be embedded in E^2 as a convex domain. This follows from the theorem that any convex domain on a manifold of non-negative curvature can be realized as a cap.

The isometric embeddings of (2.3.3) in E^2 are described by the following theorem.

Theorem 2.3.3 (Sabitov [144]). *For a locally Euclidean metric* (2.3.3) *defined in a multiply connected domain D_n to admit an isometric embedding in E^2 it is necessary and sufficient that the immersion $z = \Phi(\zeta)$ be one-sheeted on each circle Γ_j, $0 \leqslant j \leqslant n$.*

Explicit formulae for immersions of metrics defined in a simply connected domain and a series of sufficient conditions for the existence of isometric embeddings are given in [146]. For a metric defined in a multiply connected domain, there are necessary conditions for the existence of isometric immersions and a classification of cases in which the metric admits no isometric immersions into the Euclidean plane.

If a domain is two-connected, then it admits an isometric immersion in E^3.

Theorem 2.3.4 (Sabitov [143]). *Let the map $z = \Phi(\eta)$ be single-valued in $\overline{D}_n$ (that is, $\overline{D}_n$ with the metric* (2.3.1) *of class C^m can be isometrically immersed in E^2). Then the domain $\overline{D}_n$ with the metric* (2.3.1) *can be realized in E^3 as a smooth developable surface of class C^∞ in D_n and of class C^m in $\overline{D}_n$.*

Corollary. *The disc $\overline{D}_1$ with locally Euclidean metric* (2.3.1) *admits an isometric embedding in E^3 in the form of a developable surface.*

Theorem 2.3.5 (Sabitov [143]). *Suppose that the map $z = \Phi(\eta)$ is not single-valued in a domain $\overline{D}_2$ (that is, in an annulus) with locally Euclidean metric* (2.3.3) *of class C^m. Then $\overline{D}_2$ can be isometrically immersed into E^3 as a smooth developable surface of class C^m.*

There is an example of a locally Euclidean metric defined in a three-connected domain that is not immersible into E^3 in the form of a C^1 surface with Pogorelov-bounded extrinsic curvature [143]. In [169] one can find an explicit representation of the solutions of the equation

$$z_{xx}z_{yy} - z_{xy}^2 = 0$$

satisfying the condition $z_{xx} \neq 0$. This gives an explicit representation of the surfaces (without planar points) of zero Gaussian curvature in three-dimensional Euclidean space.

As is known, there are five topologically distinct types of locally Euclidean two-dimensional complete manifolds: plane, cylinder, torus, Möbius band, and Klein bottle [134].

Among these manifolds only the plane and cylinder can be regularly and isometrically immersed into E^3. One can easily immerse the torus with Euclidean metric into E^4. The parametric equations of the torus with locally Euclidean metric $ds^2 = du^2 + dv^2$ in E^4 are of the form $x_1 = \cos u$, $x_2 = \sin u$, $x_3 = \cos v$, $x_4 = \sin v$. An isometric immersion of the Klein bottle into E^4 was constructed by

Tompkins [165] (another example of immersion of the Klein bottle was constructed by Ivanov [93]). Blanuša [22] constructed an immersion of the Möbius band into E^4.

The immersion of the Klein bottle into E^4 suggested by Tompkins is determined by the following parametric equations:

$$x_1 = \cos u \cos v, \qquad x_2 = \sin u \cos v,$$
$$x_3 = 2\cos\frac{u}{2}\sin v, \qquad x_4 = 2\sin\frac{u}{2}\sin v.$$

This surface has a self-intersection curve (the circle $x_1^2 + x_2^2 = 1$, $x_3 = x_4 = 0$). If we introduce the parameters

$$U = u, \qquad V = \int \sqrt{1 + 3\cos^2 v}\, dv,$$

then the intrinsic metric of the immersion is determined by the line element $ds^2 = dU^2 + dV^2$, that is, it is locally Euclidean.

The immersion of the Klein bottle suggested by Ivanov is determined by the following parametric equations:

$$x_1 = \cos u, \quad x_2 = \sin u, \quad x_3 = a\,\sin\left(v + \frac{u}{2}\right), \quad x_4 = b\,\sin 2\left(v + \frac{u}{2}\right).$$

A representation of the surface thus obtained can be constructed as follows. For a fixed value of u one obtains a Bernoulli lemniscate located in a plane parallel to Ox_3x_4 that passes through the point $(\cos u, \sin u, 0, 0)$. As u varies from 0 to 2π, this lemniscate with given direction of circuit makes a rotation through the angle π, and then coincides with the lemniscate corresponding to the value $u = 0$. It is clear that the superposed lemniscates are traversed in opposite directions. Thus, the immersion is topologically equivalent to the Klein bottle. Introducing the parameters

$$U = u, \qquad V = \int \sqrt{2\alpha(\widetilde{v})}\, d\widetilde{v},$$

where

$$\widetilde{v} = v + \frac{u}{2}, \qquad \alpha(\widetilde{v}) = \frac{a^2}{4}\cos^2\widetilde{v} + b^2\cos^2 2\widetilde{v},$$

we see that the intrinsic metric of the immersion is determined by the line element $ds^2 = dU^2 + dV^2$, and hence is locally Euclidean.

Blanuša [22] suggested an isometric immersion of the Möbius band with locally Euclidean metric into E^4. We proceed to the description of this immersion.

Blanuša first constructs a surface S_4 in E^4 homeomorphic to the Möbius band. For this purpose, a surface S_5 in E^4 is determined by the following parametric equations:

$$x_1 = \rho\cos\frac{u}{2}, \; x_2 = \rho\sin\frac{u}{2}, \; x_3 = \frac{1}{2}\sqrt{4-\rho^2}\,\cos u, \; x_4 = \frac{1}{2}\sqrt{4-\rho^2}\,\sin u, \; x_5 = f(\rho),$$

where ρ ranges over the interval $(-R, R)$, $0 < R \leqslant 2$, and u over the closed interval $[0, 2\pi]$. The function $f(\rho)$ is even, $f(\rho) \to \infty$ as $\rho \to R$, and the derivative $f'(\rho)$ is continuous on the interval $(-R, R)$ and positive on $(0, R)$.

The surface S_5 is homeomorphic to the Möbius band. Let us prove this fact. We consider the infinite strip $\Pi = \{0 \leqslant \xi \leqslant 2\pi,\ -\infty < \eta < \infty\}$, and define the map of S_5 onto Π by the relations $u = \xi$ and $\rho = \varphi(\eta)$, where φ is the inverse function for

$$\eta = \int_0^\rho \sqrt{1 + f'^2(t)}\, dt.$$

Under the above map, the point of S_5 with coordinates

$$x_1 = \rho, \quad x_2 = 0, \quad x_3 = \frac{1}{2}\sqrt{4-\rho^2}, \quad x_4 = \frac{1}{2}\sqrt{4-\rho^2}, \quad x_5 = f(\rho)$$

corresponds to the points $(0,\eta)$ and $(2\pi,-\eta)$ of Π. Thus, S_5 is homeomorphic to the strip Π with points of the form $(0,\eta)$ and $(2\pi,-\eta)$ identified, that is, S_5 is homeomorphic to the Möbius band.

Obviously, the projection S_4 of the surface S_5 on the hyperplane (x_1,x_2,x_3,x_4) is also homeomorphic to the Möbius band. Hence, the surface S defined by the parametric equations

$$x_1 = \rho\cos\left[\frac{u}{2} + \frac{1}{2}F(\rho) - \frac{2}{R^2-\rho^2}\right], \qquad x_2 = \rho\sin\left[\frac{u}{2} + \frac{1}{2}F(\rho) - \frac{2}{R^2-\rho^2}\right],$$
$$x_3 = \frac{\sqrt{4-\rho^2}}{2}\cos[u + F(\rho)], \qquad x_4 = \frac{\sqrt{4-\rho^2}}{2}\sin[u + F(\rho)],$$

where

$$F(\rho) = \ln(R^2-\rho^2) + \frac{R^2}{R^2-\rho^2}\,,$$

is homeomorphic to the Möbius band (the surface S is homeomorphic to S_4). The line element of the surface S is

$$ds^2 = \Phi^2(\rho)\, d\rho^2 + [du + \Psi(\rho)\, d\rho]^2,$$

where

$$\Phi(\rho) = 1 + \frac{\rho^2}{4(4-\rho^2)} + \frac{16\rho^4 - 4\rho^6}{(R^2-\rho^2)^4}\,, \qquad \Psi(\rho) = F' - \frac{2\rho^3}{(R^2-\rho^2)^2}\,.$$

If one writes

$$dU = \Phi(\rho)\, d\rho, \qquad dV = du + \Psi(\rho)\, d\rho,$$

then the line element becomes $ds^2 = dU^2 + dV^2$, that is, the metric of the surface S is locally Euclidean. One can readily see that S is a complete surface [134].

In [41] Bushmelev gave isometric embeddings of the complete Möbius band and of the Klein bottle in E^4 as C^∞ surfaces. The problem of whether or not they can be embedded in the class of analytic surfaces is still open.

Let us consider an isometric immersion of a locally Euclidean metric in the Euclidean space E^4,

$$f\colon U \subset E^2 \to E^4.$$

By the *first normal space* $N_1^f(P)$ of the immersion at a point P one means the subspace of the normal space $N_P F$, where $F = f(U)$, generated by the vectors of the second fundamental form of F at the point P,

$$\alpha_f\colon T_P F \times T_P F \to N_P F.$$

Let us consider the case in which the dimension of the first normal space is constant, that is, $\dim N_1^f = \text{const}$.

Proposition 2.3.1 (Dajczer [57]). *Let h_1 and h_2 be real functions defined on a simply connected domain $U \subset E^2$ and such that*

$$h_1(u_1, u_2) = \Phi_1(u_1 + u_2) + \Phi_2(u_1 - u_2),$$
$$h_2(u_1, u_2) = -\Phi_1(u_1 + u_2) - \Phi_2(u_1 - u_2), \tag{2.3.4}$$

where Φ_1 and Φ_2 are some smooth functions of a single variable. Then for any solution (υ_1, υ_2) in U of the linear system of differential equations

$$\frac{\partial \upsilon_1}{\partial \upsilon_2} = h_2 \upsilon_2,$$
$$\frac{\partial \upsilon_2}{\partial \upsilon_1} = h_1 \upsilon_1 \tag{2.3.5}$$

such that $\upsilon_i \neq 0$ everywhere, $1 \leqslant i \leqslant 2$, there is an immersion $f\colon U \to E^4$ with flat normal connection and with $\dim N_1^f = 2$, and the induced metric

$$ds^2 = \upsilon_1^2\, du_1^2 + \upsilon_2^2\, du_2^2$$

is flat.

The converse is also true. Corresponding to any isometric immersion of a plane domain

$$f\colon U \subset E^2 \to E^4$$

with flat normal connection and with $\dim N_1^f = 2$ is a quadruple of smooth functions of the form $(\upsilon_1, \upsilon_2, h_1, h_2)$, where h_1 and h_2 are defined by (2.3.4) *and (υ_1, υ_2) is a solution of the system* (2.3.5).

We say that $f\colon U \subset E^2 \to E^4$ is a *composition* if there are isometric immersions $g\colon U \subset E^2 \to V \subset E^3$ and $h\colon V \subset E^3 \to E^4$ such that $f = h \circ g$. Let $\dim N_1^f = 1$. Then the normal connection is flat. In this case, an isometric immersion is a composition.

Theorem 2.3.6 (Dajczer, Tojeiro [55]). *Let $f\colon U \subset E^2 \to E^4$ be an isometric immersion of a plane domain with $\dim N_1^f = 1$. Then f is a composition, $f = h \circ g$, where either $g\colon U \subset E^2 \to V \subset E^3$ or $h\colon V \subset E^3 \to E^4$ is a totally geodesic immersion.*

All flat surfaces in E^4 that are not compositions were described in [60]. We first formulate the result for flat surfaces with flat normal connection. Let $\xi\colon U \subset E^2 \to S^3(1) \subset E^4$ be an isometric immersion into a sphere of unit radius, let $\widehat{\xi}\colon U \to S^3(1)$ be the polar map that sends any point U to a unit normal $\widehat{\xi} \in E^4$ of the surface $\xi(U)$, and let B be the second fundamental form of ξ with respect to the normal $\widehat{\xi}$.

Theorem 2.3.7 (Do Carmo and Dajczer [60]). *Let an isometric immersion $\xi\colon U \subset E^2 \to S^3(1) \subset E^4$ be given, and consider a pair of unit vector fields $\upsilon, \omega \in TU$ such that the following conditions hold along U:*

1) *ω is a constant vector field;*
2) $\langle \upsilon, B\omega \rangle = 0$;
3) $\langle \upsilon, \omega \rangle \neq 0$.

Let γ be an arbitrary smooth function on U satisfying the differential equation

$$\mathrm{Hess}_\gamma(\upsilon,\omega) + \gamma\,\langle \upsilon,\omega\rangle = 0,$$

and define

$$\Omega = \frac{\mathrm{Hess}_\gamma(\omega,\omega) + \gamma\ \langle \omega,\omega\rangle}{\langle B\omega,\omega\rangle}\,.$$

Let $f\colon U\subset E^2\to E^4$ be

$$f = \gamma\xi + \xi_*\,\mathrm{grad}\,\gamma + \Omega\widehat{\xi}. \tag{2.3.6}$$

Then at any regular point f is an isometric immersion of a plane domain with flat normal connection, $\dim N_1^f = 2$, and f is nowhere a composition. Conversely, let $f\colon U\subset E^2\to E^4$ be an isometric immersion with flat normal connection, let $\dim N_1^f = 2$, and let f be nowhere a composition. Then there exist an isometric immersion $\xi\colon U\subset E^2\to S^3(1)\subset E^4$ and unit vector fields $\upsilon,\omega\in TU$ satisfying conditions 1)–3) *such that f is defined by the formula* (2.3.6), *where $\gamma = \langle f,\xi\rangle$ is the support function.*

The following assertion holds for flat immersions that are compositions.

Theorem 2.3.8 (Dajczer and Tojeiro [57]). *Let $f\colon U\subset E^2\to E^4$ be an isometric immersion with flat normal connection and let $\dim N_1^f = 2$. Then the following assertions are equivalent*:

1) *f is a composition*;
2) *the four functions $(\upsilon_1,\upsilon_2,h_1,h_2)$ associated with f are equal to either*

$$(f_1(u_1)+\lambda f_2(u_2),\ f_2'(u_2),\ 0,\ \lambda)$$

or

$$(g_1'(u_1),\ \mu g_1(u_1)+g_2(u_2),\ \mu,\ 0)\,,$$

where $f_1(u_1)$, $f_2(u_2)$, $g_1(u_1)$, and $g_2(u_2)$ are smooth functions of a single variable and $\lambda,\mu\in\mathbb{R}$;

3) *$f = h\circ g$, where g is of the form*

$$g(s,t) = c(s) + t\,(An(s)+\upsilon_0)\,, \qquad A\in\mathbb{R},\quad 0\neq\upsilon_0\in\mathbb{R}^3.$$

Here $c(s)$ stands for a curve parametrized by the arclength with non-zero curvature and with normal vector $n(s)$ orthogonal to υ_0, and h is a right cylinder in which the plane orthogonal to υ_0 is a Euclidean factor [57].

We say that an immersion $\xi\colon U\subset E^2\to S^3(1)$ is a *type C* surface if there are two regular linearly independent vector fields on U such that

$$\nabla'_V W = 0, \qquad \langle BV,W\rangle = 0, \qquad \langle BW,W\rangle \neq 0,$$

where ∇' is the induced connection on $\xi(U)$.

Let $\gamma = \langle f,\xi\rangle$, and let a linear map $P\colon TU\to TU$ be such that

$$\langle PZ,W\rangle = \gamma\,\langle Z,W\rangle + \mathrm{Hess}_\gamma(Z,W), \qquad Z,\ W\in TU.$$

Theorem 2.3.9 (Do Carmo, Dajczer [60]). *Let $\xi \colon U \subset E^2 \to S^3(1) \subset E^4$ be a type C surface with $\langle V, W\rangle \neq 0$. Let γ be a smooth function satisfying the condition $\langle PV, W\rangle = 0$. We write*

$$\Omega = \frac{\langle PW, W\rangle}{\langle BW, W\rangle}.$$

Let $f \colon U \to E^4$ be

$$f = \gamma\xi + \xi_* \nabla'\gamma + \Omega\widehat{\xi}. \tag{2.3.7}$$

Then f is a flat immersion with $\dim N_1^f = 2$ which is nowhere a composition; here ξ stands for a unit normal vector field for which the rank of the Weingarten transformation A_ξ is equal to 1. Conversely, if $f \colon U \subset E^2 \to E^4$ is a flat immersion that is nowhere a composition and if $\dim N_1^f = 2$, then there is a type C surface $\xi \colon U \to S^3(1)$ such that f is given by (2.3.7) with $\gamma = \langle f, \xi\rangle$.

The condition $R_f^\perp \neq 0$ for the normal connection is equivalent to the condition $\nabla'_W W \neq 0$.

The regularity of an immersion is determined by the conditions

$$\langle PV, V\rangle \neq \Omega \langle BV, V\rangle, \qquad \langle B(\nabla'\gamma) + \nabla'\Omega, W\rangle \neq 0.$$

Conjecture. Let F be a compact regular surface in E^4 with zero Gaussian curvature and with flat normal connection. Then this surface is either a flat torus in the spherical space S^3 or the Cartesian product of two closed curves that belong to orthogonal two-dimensional planes.

2.4. Isometric immersions of domains of the Lobachevsky plane. Not only the Lobachevsky plane itself but also a half-plane cannot be isometrically immersed into the Euclidean space E^3 in the class of regular surfaces. In the class of C^4 regular surfaces this fact was proved by Efimov [62], and in the C^2 regularity class it was proved in [171]. Then a natural question arises: What domains of the Lobachevsky plane are immersible into E^3?

Minding found the surfaces of constant negative curvature that are realizations of strips of the Lobachevsky plane lying between two equidistant curves. Beltrami constructed a regular realization of a 'full horocycle', that is, a domain bounded by a horocycle of the Lobachevsky plane. A half of a pseudo-sphere defines a regular isometric immersion of the full horocycle.

Isometric immersions of an infinite polygon with finitely or infinitely many sides that do not contain a half-plane are considered in Poznyak's paper [135]. By an *infinite polygon* one means an intersection of closed half-planes whose boundaries are straight lines on L^2 that have no common points. The set of polygons under consideration contains no half-planes, and each side of any polygon in this set has a parallel side, which is said to be *neighbouring*. Two classes of such polygons are introduced, M_1 and M_2. The first class M_1 consists of all polygons for which there is a horocycle O in the plane such that the greatest lower bound of the lengths of the orthogonal projections of the sides on this horocycle is positive. The other class M_2 consists all infinite polygons for which the greatest lower bound of the lengths of the orthogonal projections of the sides on one of the sides is positive.

Theorem 2.4.1 (Poznyak [135]). *Any polygon belonging to one of the classes M_1 and M_2 can be isometrically immersed in E^3.*

The proof of this theorem is based on a geometric idea which will be clarified for polygons of the first class. One must find a horocycle O_α that is equidistant from the horocycle O and cuts off all vertices of the polygon. As is known, the universal covering of the pseudo-sphere (to be more precise, of a half of the pseudo-sphere) gives an isometric immersion of some full horocycle for which one can take O_α. Therefore, one has a regular Chebyshev net on a part of the polygon that is located inside O_α. It remains to extend this net to the parts outside O_α. We join the vertices of the polygon to the point at infinity Q of the full horocycle O_α by geodesic curves a_i and construct an infinite equidistant strip along a_i of width $2h$. For some width $2h$ the part of the polygon that is outside the full horocycle O_α belongs completely to the strip, in which one can introduce a semigeodesic system of coordinates and write out the system of equations of immersion in this system of coordinates. The initial data for the solution of this system are given on a_i in such a way that on the part of a_i outside O_α these data extend the values of the desired functions on the part of a_i inside O_α, while the latter values correspond to an isometric immersion of the pseudo-sphere.

After this one can apply the existence and uniqueness theorem proved by Poznyak for the solutions of the system of immersion in a strip. Thus, one can obtain immersions of polygons of class C^∞ [10].

The following non-compact domains of the Lobachevsky plane admit a regular realization [100], [101]:

a) an expanding strip with geodesic base;
b) a convex domain containing two full horocycles;
c) a convex domain whose boundary admits an arbitrary (finite) order of tangency with the absolute;
d) a new class of infinite convex polygons.

Let

$$\varphi(y) = \lambda\sqrt{y^2 + a^2} - \left(2\lambda + \nu + \frac{1}{6}\right)$$

be a function defining an expanding strip, where

$$a = \text{const} > 11, \qquad \lambda = \text{const} > \frac{3}{a-2},$$

y stands for the arclength on the geodesic curve that is the base of the strip, and $\nu = \ln(29\lambda^2 + 24\lambda + 3)$.

Theorem 2.4.2 (Kaidasov and Shikin [100], [101]). *The expanding strip $\pi(\varphi)$ with the function $\varphi(y)$ can be realized in E^3 as a regular* (C^3 *regular*) *surface.*

Theorem 2.4.3 (Kaidasov and Shikin [100], [101]). *Any two* (*full*) *horocycles can be included in a convex domain that is* (*regularly and isometrically*) *immersible into E^3.*

Theorem 2.4.4 (Kaidasov and Shikin [100], [101]). *On the Lobachevsky plane there is an unbounded convex domain Q with the following properties*:

1) *Q contains a full horocycle*;
2) *no full horocycle contains all points of Q*;
3) *Q contains no complete geodesic curve*;
4) *Q can be realized in E^3 as a regular (C^3) surface.*

It was proved that any infinite convex polygon belonging to a certain new class can be isometrically immersed into E^3. Such a polygon is characterized by the condition that its sides are tangent to the boundary of an expanding strip whose base is one of the sides of the polygon. To prove that the polygons of the above type can indeed be immersed into E^3, a solution of the main system is first constructed in a special expanding strip and then is extended to a domain containing the polygon by using the Poznyak approach.

Blanuša obtained specific parametric equations of a surface of class C^∞ without self-intersections in E^6 such that the intrinsic metric of this surface coincides with the metric of the Lobachevsky plane. Let

$$\psi_1(u) = e^{2[(|u|+1)/2]+5}, \qquad \psi_2(u) = e^{2[|u|/2]+6},$$

where $[\,\cdot\,]$ stands for the integral part of the bracketed expression. We set

$$A = \int_0^1 \sin\pi\,\xi\, e^{-1/\sin^2\pi\xi}\, d\xi,$$
$$\varphi_1(u) = \left(\frac{1}{A}\int_0^{u+1} \sin\pi\,\xi\, e^{-1/\sin^2\pi\,\xi}\, d\xi\right)^{1/2},$$
$$\varphi_2(u) = \left(\frac{1}{A}\int_0^{u} \sin\pi\,\xi\, e^{-1/\sin^2\pi\,\xi}\, d\xi\right)^{1/2},$$
$$f_1(u) = \frac{\varphi_1(u)}{\psi_1(u)}\sinh u, \quad f_2(u) = \frac{\varphi_2(u)}{\psi_2(u)}\sinh u.$$

Let x_i ($i = 1, \dots, 6$) be the Cartesian coordinates in E^6. Then the embedding in E^6 of the Lobachevsky plane with the line element

$$ds^2 = du^2 + \sinh^2 u\, dv^2$$

is given by the following parametric equations:

$$\begin{aligned}
x_1 &= \int_0^u \sqrt{1 - f_1'^2(\xi) - f_2'^2(\xi)}\, d\xi,\\
x_2 &= v,\\
x_3 &= f_1(u)\cos(v\,\psi_1(u)),\\
x_4 &= f_1(u)\sin(v\,\psi_1(u)),\\
x_5 &= f_2(u)\cos(v\,\psi_2(u)),\\
x_6 &= f_2(u)\sin(v\,\psi_2(u)).
\end{aligned}$$

The fact that the line element coincides with that of the Lobachevsky plane is proved by subtle analytic arguments [23].

Theorem 2.4.5 (Rozendorn [139]). *The Lobachevsky plane admits a regular isometric immersion into* E^5.

Theorem 2.4.6 (Sabitov [145]). *The Lobachevsky plane can be isometrically immersed into* E^4 *as a piecewise analytic surface that is smooth in the large of class* $C^{0,1}$. *However, at the same time the Lobachevsky plane admits no isometric immersion into* E^4 *as a generalized surface of revolution.*

Rozendorn [140] proved that the Lobachevsky plane admits no isometric immersion into E^4 as a regular helicoidal surface. In [114] this result was generalized to helicoidal immersions of L^2 into finite-dimensional Euclidean spaces E^n, and in [112] to helicoidal immersions of L^l into E^n.

In [23], [24], and [26] Blanuša gave the following explicit C^∞ isometric embeddings:

1) the Lobachevsky plane in the spherical space S^8;
2) a cylinder with hyperbolic metric of constant Gaussian curvature -1 in the Euclidean space E^7 and in the spherical space S^8;
3) an infinite Möbius band with hyperbolic metric in the Euclidean space E^8 and in the spherical space S^{10} (see also [138]).

In [73], using the complex structure induced by the second fundamental form, a conformal representation of intrinsically flat surfaces (that is, surfaces with zero Gaussian curvature) in the three-dimensional hyperbolic space is given. The problem is equivalent to finding the solutions of the Monge–Ampère equation

$$z_{xx}z_{yy} - z_{xy}^2 = 1.$$

2.5. Isometric immersions with constant extrinsic geometry. The problem of the relationship between the group of motions of Lobachevsky space and the group of motions of Euclidean space is of interest. A natural formulation of this question is the problem of whether or not there are isometric immersions of Lobachevsky spaces in a Euclidean space under which some group of motions of the Lobachevsky space is induced by the motions of the enveloping Euclidean space. Apparently, Bieberbach [17] was the first who considered this problem. He constructed an isometric immersion of the Lobachevsky plane into a Hilbert space under which any intrinsic motion of the Lobachevsky plane can be realized by means of a corresponding motion of the Hilbert space (examples of immersions into E^3 of this type for metrics of zero curvature and of constant positive curvature are given by a plane and a sphere).

Bieberbach constructed an analytic embedding of L^2 in a Hilbert space. This embedding is given by explicit formulae. Before writing them out, we introduce some auxiliary notation. Let $F_m(z)$ be a finite or infinite set of functions of a complex variable $z = u + iv$, not necessarily analytic. We write

$$x_{2m-1} = \operatorname{Re} F_m(z), \quad x_{2m} = \operatorname{Im} F_m(z), \qquad m = 1, \dots, p, \tag{2.5.1}$$

assuming that x_j are the Cartesian coordinates in E^N, $N \geqslant 2p$.

The Bieberbach construction looks very simple. Namely, in (2.5.1) one can set $F_m(z) = m^{-1/2} z^m$ and $p = N = \infty$. The immediate calculation shows that for $p = |z| < 1$ we obtain an embedding in E^∞ for the metric $ds^2 = (1-p^2)^{-2}(du^2 + dv^2)$, and, as is known, this is one of representations of the metric in L^2. The conditions $x_1 = u$ and $x_2 = v$ are satisfied, and the surface admits the natural one-sheeted projection onto the disc $p < 1$ of the plane x_1, x_2.

Bieberbach proved that his surface of the form (2.5.1) belongs to no finite-dimensional subspace and has the following interesting property: the entire group of motions is induced on this surface by motions of the enveloping space. In the same paper Bieberbach presented a proof (due to Schmidt) of the following fact: in E^N $(N < \infty)$ the group of all motions of an immersed surface L^2 cannot be induced by the motions of E^N (although at that time it was not known whether such immersions really exist).

In [18], an isometric embedding of an n-dimensional Lobachevsky space in a Hilbert space is given. However, the problem of whether or not all motions of this embedded Lobachevsky space are induced by motions of the enveloping Hilbert space was not studied there. In my opinion, this problem remains open. For the subsequent investigation of these problems, see the paper of Kadomtsev [97].

Definition. A surface $\Omega \subset E^n$ is said to be a *surface of revolution with a pole* at a point $M \in \Omega$ if for any two geodesic curves l_1 and l_2 emanating from the point M there is a motion G of the space E^n such that

1) $G(M) = M$;
2) $G(\Omega) = \Omega$;
3) $G(l_1) = l_2$.

No part of the Lobachevsky plane can be immersed into E^3 as a surface of revolution with a pole. However, such an immersion becomes possible already into E^4.

Theorem 2.5.1 (Kadomtsev [97]). *The Lobachevsky plane L^2 cannot be immersed in the large into any finite-dimensional Euclidean space E^n as a surface of revolution with a pole.*

The complete Lobachevsky space L^m $(m \geqslant 2)$ admits no immersion into a finite-dimensional Euclidean space E^n under which any intrinsic rotation of L^m about a chosen point $M \in L^m$ is induced by some motion of the enveloping space E^m.

In particular, it follows from Theorem 2.5.1 that it is impossible to find an immersion of the Lobachevsky plane L^2 into a finite-dimensional Euclidean space under which any intrinsic motion of L^2 can be induced by some motion of the enveloping space.

Definition. A surface $\Omega \subset E^n$ of constant curvature is said to be a *surface with parallel displacement* if for any two points P and Q of Ω there is a motion G of E^n such that 1) $G(\Omega) = \Omega$, 2) $G(l) = l$, 3) $G(P) = Q$, where l is the geodesic curve of Ω that passes through the points P and Q.

Theorem 2.5.2 (Kadomtsev [97], [99]). *The Lobachevsky plane L^2 cannot be immersed into any finite-dimensional Euclidean space as a surface with parallel displacement.*

Theorem 2.5.3 (Kadomtsev [98]). *The complete Lobachevsky plane cannot be C^0 immersed into any finite-dimensional Euclidean space in such a way that some continuous non-trivial subgroup of the group of motions of L^2 is induced by motions of the Euclidean space.*

We note that the central place in the proof of these statements is occupied by the proof of a certain auxiliary assertion, which is, however, of independent interest. The heart of the problem is as follows. Let L be a curve of class C^0 in a Euclidean space with the following property: for any two points of L there are neighbourhoods of these points congruent to each other (examples of such curves in E^3 are given by a line, a circle, and a helical curve); then the curve L is analytic.

Let us briefly sketch the proof of Theorem 2.5.3. Suppose the contrary. We thus assume that the above immersion is possible. Then the metric of the corresponding surface can be represented in the form

$$dl^2 = ds^2 + H^2(s)\,dt^2,$$

where the t-coordinate lines are at the same time trajectories of both a one-parameter group of motions of the Lobachevsky plane and of a one-parameter group of motions of the enveloping Euclidean space. Using the form of the motions of the enveloping space, one can obtain the estimate $|H(s)| \leqslant Cs^2$ for the function $H(s)$ for all sufficiently large s. On the other hand, the function $H(s)$ obviously has exponential growth. The contradiction thus obtained proves the assertion (see also [10]).

It is natural to study the surfaces for which any intrinsic motion is induced by a motion of the enveloping Euclidean space.

A two-dimensional surface $F^2 \subset E^n$ is called a *surface of constant extrinsic geometry* if for any two points $X \in F^2$ and $Y \in F^2$ there are neighbourhoods $U(X)$ and $U(Y)$ on the surface that are congruent in E^n.

Theorem 2.5.4 (Garibe [74]). *The only two-dimensional surfaces of constant extrinsic geometry in E^4 of regularity class C^3 are the planes, the spheres, the generalized Clifford tori, and cylindrical surfaces whose directrices are helical curves in some E^3 and whose rectilinear generators are orthogonal to E^3.*

Theorem 2.5.5 (Garibe [75]). *If a two-dimensional surface in E^5 with positive Gaussian curvature is of constant extrinsic geometry, then it is either a sphere or a Veronese surface.*

Theorem 2.5.6 (Garibe [75]). *Any two-dimensional surface with constant extrinsic geometry and zero Gaussian curvature in E^5 can be represented in the form*

$$x_j = \sum_{i=1}^{2}\Bigl[\cos\gamma_i v \sum_{k=1}^{2}(a^i_{j,2k-1}\cos\delta_k u + a^i_{j,2k}\sin\delta_k u) + \sin\gamma_i v \sum_{k=1}^{2}(a^i_{j,2k+3}\cos\delta_k u + a^i_{j,2k+4}\sin\delta_k u)\Bigr],$$

$j = 1,2,3,4$, $x_5 = C_1u + C_2v$, where a^i_{jk}, C_1, and C_2 are determined by the parameters of the ellipse of normal curvature and by the torsion coefficients of the surface.

It is natural to find all two-dimensional and multidimensional surfaces of constant sectional curvature for which all intrinsic motions are induced by the motions of the enveloping space. This problem is related to minimal immersions of the multidimensional sphere $S^n_{K(s)}$ into $S^{m(s)}_1$, where $K(s) = n/s(s+n-1)$ and

$$m(s) = (2s+n-1)\frac{(s+n-2)!}{s!\,(n-1)!} - 1.$$

These immersions are constructed by means of spherical harmonics of order s, that is, the restrictions to S^n_1 of homogeneous polynomials $P(x_0, x_1, \dots, x_n)$ of degree s in E^{n+1} that satisfy the equation

$$\sum_i \frac{\partial^2 P}{\partial x_i^2} = 0.$$

The dimension of the space of spherical harmonics of order s is equal to $m(s)+1$, and if $f_0, \dots, f_m$ is an orthonormal basis of this space, then the desired isometric immersion $M \to E^{m+1}$ is

$$\psi(p) = (f_0(p), \dots, f_m(p)).$$

2.6. Immersion of the projective plane. It follows from Theorem 2.2.5 that the only isometric regular immersion of the standard sphere $S^2(1)$ into $S^3(1)$ is a totally geodesic sphere. On the other hand, the Clifford flat torus T^3 can be isometrically embedded in the sphere $S^5 \subset E^6$, and hence the immersed sphere $S^2(R) \subset T^3$ of radius R can be isometrically immersed into $S^5(1)$. Isometric non-standard immersions $f\colon S^2(R) \to S^4$ of spheres are found for $0 < R < 4$ in [72], and these immersions are embeddings for $R < 1.6$.

The real projective plane $\mathbb{R}P^2$ admits a metric of constant positive curvature. We claim that it admits no isometric embedding of class C^2 into the Euclidean space E^4.

As shown by Whitney, the real projective plane admits no differentiable embedding in E^4 with normal vector field without singularities. Therefore, it suffices to prove that, under an isometric C^2 immersion of a two-dimensional Riemannian manifold of positive curvature into E^4, a normal vector field without singularities always exists.

Let $f\colon V \to E^4$ be the desired immersion. We denote by T_υ the tangent plane of V at a point $\upsilon \in V$ and by N_υ the normal plane, and we construct the 'second fundamental form' $(d^2_\upsilon f, \xi)$ for an arbitrary vector $\xi \in N_\upsilon$. This is a quadratic form on T_υ with trace $\sigma(\xi)$ and discriminant $\delta(\xi)$. The trace σ is a linear form on N_υ and the discriminant δ is a quadratic form on N_υ. This form has its own trace which is equal to the curvature of the manifold V at the point υ by the Gauss formula, and therefore the latter trace is positive. Since a quadratic form with positive trace takes positive values, it follows that $\delta(\xi) > 0$ for some $\xi \in N_v$. Since a binary quadratic form with positive discriminant has non-zero trace, it follows that $\sigma(\xi) \neq 0$ for this ξ. Thus, σ is a non-zero linear form on N_v. This form depends continuously on v, and hence it has a corresponding dual vector field, which is just the desired normal vector field without singularities [80].

An isometric embedding of $\mathbb{R}P^2$ of curvature $\frac{1}{r^2}$ in $S^4(\rho) \subset E^5$, where $\rho = \frac{r}{\sqrt{3}}$, is given by the following radius vector [21]:

$$x_1 = \frac{r}{2}\sin^2 u \sin 2v, \quad x_2 = \frac{r}{2}\sin^2 u \cos 2v,$$
$$x_3 = \frac{r}{2}\sin^2 u \sin v, \quad x_4 = \frac{r}{2}\sin^2 u \cos v, \quad x_5 = \frac{\rho}{4\sqrt{3}}(1 + 3\cos 2u).$$

This is the Veronese surface. In [21] it is shown that a real projective space of constant Gaussian curvature $k = \frac{1}{r^2}$ can be isometrically immersed into the sphere $S^4(\rho)$, $\rho > r/3$, in the real projective space $\mathbb{R}P^4(\rho)$ of constant curvature $K = \frac{1}{\rho^2}$, that is,

$$k \geqslant \frac{K}{3}\,.$$

Does there exist an immersion for $k < \frac{K}{3}$?

There is also an immersion of $\mathbb{R}P^2$ into the Euclidean space E^4 and into the Lobachevsky space L^4.

CHAPTER 3

ISOMETRIC IMMERSIONS OF MULTIDIMENSIONAL SPACE FORMS

3.1. Isometric immersions of space forms by submanifolds of zero extrinsic curvature. Let F^l be a regular submanifold of class C^2 with zero sectional curvature in a Euclidean space E^{l+p}. Then F^l is locally isometric to a Euclidean space.

Let F^l be a regular submanifold of class C^2 in a Riemannian space M^n, let $Q \in F^l$ be a point on the submanifold, let ξ be a unit normal vector field in a neighbourhood of the point Q, let X and Y be vector fields tangent to the manifold, let $\overline{\nabla}$ be the Levi-Civita connection on the Riemannian manifold M^n, let ∇ be the induced connection on F^l, let $\nabla^\perp$ be the connection in the normal bundle of F^l, let $\alpha(X, Y)$ be the vector second fundamental form of F^l, and let A_ξ be the linear transformation of $T_Q F^l$ corresponding to the second fundamental form with respect to the normal ξ.

Definition. The *external null-index* (or *relative nullity*) $\mu(Q)$ at a point $Q \in F^l$ is the maximal dimension of a subspace $L(Q)$ of the tangent space $T_Q F^l$ such that

$$A_\xi y = 0$$

for any vector $y \in L(Q)$ and for any normal $\xi \in N_Q F^l$ at this point. A submanifold is said to be k-strongly parabolic if the external null-index μ is not less than k for any point of the submanifold.

Chern and Kuiper [48] proved that for a submanifold F^l of zero extrinsic curvature in a Riemannian space M^{l+p} the external null-index satisfies the inequality

$$\mu \geqslant l - p \tag{3.1.1}$$

if the codimension is p, $p < l$, and $\mu > 0$.

If the null-index μ is constant in a neighbourhood of a point Q, then on a surface of class C^3 one has a differentiable null-distribution $L(Q)$. However, without additional assumptions on a surface in a Riemannian space, this distribution can be even non-integrable.

Suppose that the curvature tensor of the enveloping Riemannian manifold M^n satisfies the following condition along the submanifold F^l:

$$R(X,Y)Z^{\perp} = 0, \tag{3.1.2}$$

where R stands for the curvature operator, X, Y, and Z for vectors tangent to F^l, and $\perp$ for the orthogonal projection onto the normal space of F^l. For a space of constant curvature C one has

$$R(X,Y)Z = C(\langle X,Z\rangle Y - \langle Y,Z\rangle X),$$

and the condition (3.1.2) is satisfied along any submanifold.

Lemma 3.1.1 (Maltz [113]). *Let F^l be a regular submanifold of class C^3 in a Riemannian space M^n. Suppose that in a neighbourhood of a point $P_0 \in F^l$ the null-index satisfies the condition $\mu(Q) = k = \mathrm{const}$ and that the condition (3.1.2) holds at the points of the submanifold. In this case the null-distribution $L(Q)$ is integrable, and the leaves $SL(Q)$ are totally geodesic submanifolds of the enveloping space M^n.*

The normal space is stationary along the leaves, that is, an arbitrary normal $\xi(Q)$ at a point Q of the corresponding leaf $SL(Q)$ remains normal to the submanifold F^l after a parallel displacement with respect to the connection of the enveloping space along a path $\gamma \in SL(Q)$.

If F^l is a complete submanifold and $\mu(P_0) = \min_{Q\in F^l} \mu(Q) = k$, then the leaf $SL(P_0)$ is a complete totally geodesic submanifold.

If the enveloping space is a space of constant curvature, then the leaves are totally geodesic submanifolds of constant curvature, in particular, these are great spheres in a Euclidean space, and Lobachevsky spaces of lesser dimension in a Lobachevsky space.

Let Q be a point on a locally Euclidean submanifold F^l such that the external null-index is constant in a neighbourhood of Q. Then it follows from Lemma 3.1.1 that there is a μ-dimensional plane of the enveloping Euclidean space that passes through Q and along which the normal space is stationary.

If the submanifold F^l is complete and if a point $Q \in F^l$ is such that

$$\mu_0 = \mu_0(Q) = \min_{Q\in F^l} \mu(Q),$$

then it follows from Lemma 3.1.1 that the complete surface F^l contains a Euclidean plane E^{μ_0}.

The following assertion holds.

Lemma 3.1.2 [34]. *Let a submanifold F^l be an isometric immersion of class C^0 of a complete Riemannian manifold M^l which is a metric product $M^l = M^{l-k} \times E^k$, where E^k is a Euclidean factor. If the image of $(Q_0 \times E^k)$ contains k linearly independent lines of the Euclidean space for some point Q_0 of M^{l-k}, then the submanifold F^l is a cylinder with k-dimensional generator.*

This lemma follows from Lemma 2.1.1. By the estimate (3.1.1), Lemmas 3.1.1 and 3.1.2 imply the following result for locally Euclidean metrics.

Theorem 3.1.1 (Hartman [81]). *Let F^l be a complete regular locally Euclidean submanifold of the Euclidean space E^{l+p}, $p < l$. Then F^l is a cylinder with $(l-p)$-dimensional generator.*

For $p = 1$ this is the Hartman–Nirenberg theorem [58]. For $p = l$ the Clifford torus carries a locally Euclidean metric and is a compact submanifold. Theorem 3.1.1 implies the Tompkins theorem which states that there is no compact submanifold F^l of zero sectional curvature in E^{l+p} [48] for $p < l$.

In my opinion, it would be of interest to estimate the dimension of a Euclidean space E^{l+p} in which a flat space form M^l can be isometrically immersed in dependence on the complexity of the fundamental group $\pi_1(M^l)$.

Let F^l be an isometric immersion of the standard sphere $S^l(1)$ of unit curvature in the sphere $S^{l+p}(1)$. The submanifold F^l is also a strongly parabolic submanifold in the spherical space S^{l+p}, and if $p < l$, then it admits the same estimate for the external null-index and the same structure; one must just replace the planes of a Euclidean space by totally geodesic spheres, and if the surface F^l is complete, then it contains a totally geodesic sphere S^{l-p}.

Lemma 3.1.3 ([27], [30]). *Let F^l be a smooth surface of class C^1 and let F^l be isometric to S^l in the spherical space S^n. If F^l contains a great circle, then F^l is a totally geodesic great sphere.*

In the case $l = 2$ this is Theorem 2.2.5.

Let us now consider the general case. Let S^1 be a great circle of S^{l+p} belonging to F^l, and let F_{l-2} be the set on F^l polar to S^1 on F^l. This set belongs to the sphere S^{l+p-2} that is polar to S^1 on S^{l+p}. Let X and Y be arbitrarily chosen antipodal points of S^1, and let $Z \in F_{l-2}$. The semi-circles on S^{l+p} that pass through $Z \in F_{l-2}$ and have ends at X and Y belong to F^l and form a surface F_{l-1} isometric to S^{l-1}. We claim that F_{l-1} is a smooth surface at the point X.

The surface F^l is obtained from F_{l-1} by a rotation. Under this rotation, the points of F_{l-2} are fixed and the point X moves along S^1. From the assumption that F_{l-1} is not smooth it follows that F^l is also not smooth, which contradicts an assumption of the lemma.

Thus, the surface F_{l-1}, which is smooth at the point X, is just a great sphere S^{l-1}, and hence F^l is a great sphere S^l in S^{l+p}.

Here is an analogue of Theorem 3.1.1 for the spherical space.

Theorem 3.1.2 ([27], [30]). *Let F^l be a regular isometric immersion of class C^2 of the standard sphere of unit curvature into a spherical space $S^{l+p}(1)$. If $p < l$, then F^l is a totally geodesic great sphere.*

Theorem 3.1.2 was re-proved by Ferus in [70].

The corresponding analogue for hyperbolic spaces is as follows.

Theorem 3.1.3 (Zeghib [177]). *Let $f\colon V^l(-1) \to W^{l+p}(-1)$ be a regular isometric immersion of a compact hyperbolic space of curvature -1 into a hyperbolic space form of the same curvature. If $p < l$, then $f(V)$ is a totally geodesic submanifold.*

Since the external null-index satisfies the condition $\mu \geqslant l-p > 0$, it follows that in a neighbourhood of a point with minimum null-index μ_0 one obtains a geodesic foliation on $f(V)$ into complete leaves (Lemma 3.1.1). Zeghib also proved that there is no C^1 geodesic foliation into complete leaves on a hyperbolic manifold (not necessarily complete) of finite volume. This implies that $\mu_0 = l$, and $f(V)$ is a totally geodesic submanifold [178].

Isometric immersions of the Lobachevsky space $L^l(-1)$ into $L^{l+1}(-1)$ were studied by Ferus in [69]. These submanifolds are strongly parabolic. If the external null-index $\mu(Q)$ is equal to $l-1$ at any point, then the curvature of a curve orthogonal to the complete totally geodesic leaves of the null-distribution does not exceed 1, where the curvature is taken with respect to the metric of the surface. Moreover, the following assertion holds.

Theorem 3.1.4 (Ferus [69]). *Corresponding to any totally geodesic hyper-foliation of the Lobachevsky space L^l is an isometric immersion of L^l into L^{l+1} under which the leaves are mapped into leaves of the null-distribution of an $(l-1)$-strongly parabolic hypersurface, and the immersion has no umbilic points, that is, the external null-index is equal to $\mu(Q) = l-1$ at any point.*

Alexander and Portnoy [6] specified the extrinsic structure of an isometric immersion $H^l \to H^{l+1}$ with external null-index $\mu(Q)$ equal to $l-1$ at any point Q. Let $\overline{L}^{l+1} = L^{l+1} \cup L^l_\infty$, where L^l_∞ is the ideal boundary.

Theorem 3.1.5 (Alexander and Portnoy [6]). *Let $\eta\colon L^l \to L^{l+1}$ be an isometric immersion that is free of umbilic points and let $\{L_c\}_{c\in I}$ be a family of generators of η. Then there is a unique curve $\sigma\colon I \to L^{n+1}$ such that the generators $\{L_c\}$ are parallel in $\overline{L}^{n+1}$ along the curve σ.*

In a sense, this formulation is similar to the Hartman–Nirenberg theorem which states that an isometric immersion of a Euclidean space E^l into E^{l+1} is a cylinder with $(l-1)$-dimensional generator.

Cylindrical immersions of L^l into L^{l+p} were studied in [36] and [35] as a special case of strongly parabolic submanifolds. The following assertion holds.

Theorem 3.1.6 [35]. *Let F^l be an isometric immersion of the Lobachevsky space $L^l(-1)$ into a Lobachevsky space $L^{l+p}(-1)$ with constant external null-index μ. Let $p < l$, and let there be a surface orthogonal to the null-leaves. If a surface orthogonal to the null-leaves is totally umbilical as a submanifold of F^l (if it degenerates into a curve, then its geodesic curvature must be constant), then F^l is a cylindrical surface in the Lobachevsky space $L^{l+p}(-1)$; in the Cayley–Klein model, this surface is given by a $(k-1)$-cylinder ($k \geqslant l-p$) over a conic surface with a vertex at the absolute or outside the absolute.*

The theorem holds without the restriction on the codimension p if the isometric immersion is realized in the class of strongly k-parabolic submanifolds with constant null-index.

In [2] isometric immersions of L^l in L^{l+1} were studied for the case in which umbilic points can exist (in this case these are planar points), that is, the external null-index $\mu(Q)$ can be equal to l. To cite the theorem, we need the definition of lamination.

Definition. Let $U \subset L^n$ be an at most countable union of open connected components each of which is C^∞ foliated into complete totally geodesic hypersurfaces. We denote this foliation by F_U. The set $L^n - U$ is closed. The triple $(U, F_U, L^n - U)$ is called a C^∞ *lamination* on L.

Corresponding to any isometric immersion $f\colon L^l \to L^{l+1}$ is a lamination in which U is the set of points Q with external null-index $\mu(Q) = l - 1$, F_U is the foliation of null-distributions of the second fundamental form, and $L^n - U$ is the set of planar points with $\mu(Q) = l$. The converse is also true.

Theorem 3.1.7 (Abe and Hass [2]). *For any differentiable lamination on L^n there is a family of isometric immersions of L^n into L^{n+1} such that the induced immersion of the null-foliation into totally geodesic hypersurfaces of the set U completely coincides with the foliation F_U of the lamination, and the set of planar points coincides with the set $L^n - U$.*

A complete parametrization of the isometric immersions $L^l \to L^{l+1}$ is given in [5], where the space of isometric immersions $L^l \to L^{l+1}$ is parametrized by a family of at most countably many appropriately chosen n-rows of real-valued functions defined on an open interval. An analytic approach to this problem was realized in [86], as it was done earlier for the two-dimensional case in [88]. In this case to any isometric immersion one assigns a solution of a degenerate system of equations of Monge–Ampère type in the unit ball $B^n(1)$.

Theorem 3.1.8 (Hu [90]). *Corresponding to any smooth isometric immersion $L^n(-1) \to L^{n+1}(-1)$ is a solution of the system of equations*

$$\frac{\partial^2 u}{\partial\xi_i\,\partial\xi_j}\cdot\frac{\partial^2 u}{\partial\xi_k\,\partial\xi_l} - \frac{\partial^2 u}{\partial\xi_i\,\partial\xi_k}\cdot\frac{\partial^2 u}{\partial\xi_j\,\partial\xi_l} = 0, \tag{3.1.3}$$
$$\xi = (\xi_1, \dots, \xi_n) \in B^n(1), \quad i, j, k, l = 1, \dots, n.$$

Conversely, for any smooth solution U of the system (3.1.3) *let*

$$\begin{cases} g_{ij} = \lambda^{-4}(\lambda^2\delta_{ij} + \xi_i\xi_j), & \lambda = \sqrt{1 - \xi_1^2 - \dots - \xi_n^2}, \\ h_{ij} = \lambda^{-1}\dfrac{\partial^2 u}{\partial\xi_i\,\partial\xi_j}, & i, j = 1, \dots, n. \end{cases} \tag{3.1.4}$$

The solution U defines an isometric immersion $L^n(-1) \to L^{n+1}(-1)$ for which g and h are the first and the second fundamental forms, respectively.

3.2. Isometric immersions of space forms as submanifolds of negative extrinsic curvature. A locally isometric embedding of the Lobachevsky space L^l in the Euclidean space E^{2l-1} was given by Shur [156]. It is of the form

$$x_{2i-1} = a_i e^{y_l}\cos(y_i/a_i), \quad x_{2i} = a_i e^{y_l}\sin(y_i/a_i), \qquad 1 \leqslant i \leqslant l-1,$$
$$x_{2l-1} = \int_0^{y_l} (1 - e^{2u})^{1/2}\,du,$$

where $a_i \neq 0$ and $\sum_{i=1}^{l-1} a_i^2 = 1$.

This is a generalization of the pseudo-sphere to the multidimensional case. Cartan proved in 1919 that one cannot embed the Lobachevsky space L^l locally isometrically in E^{2l-2} [44], [111].

Theorem 3.2.1 (Otsuki [127]). *A Riemannian n-dimensional space of strictly negative sectional curvature cannot be embedded in* E^{2n-2} *locally isometrically.*

The proof of the theorem makes heavy use of the lemma formulated below. Let us agree that the indices range as follows: $1 \leqslant i, j, k, l \leqslant n$, $1 \leqslant r \leqslant n-2$, $1 \leqslant s \leqslant n-1$, $1 \leqslant m \leqslant N$, and $1 \leqslant \alpha, \beta, \gamma, \delta, \rho \leqslant 2n-2$. Let a system of quadratic equations with n unknowns be given,

$$A^m_{ij} x^i x^j = 0. \tag{3.2.1}$$

We denote by $A(x,x)$ the vector with components $A^m_{ij} x^i x^j$ and by $A(x,y)$ the vector with components $A^m_{ij} x^i y^j$, where x and y are n-dimensional vectors with components x^i and y^i, and $\langle \cdot , \cdot \rangle$ is the inner product in the Euclidean space.

Lemma 3.2.1. *If in the system of* N *equations* (3.2.1) *one has* $N \leqslant n-1$ *and* $\langle A(x,x), A(y,y)\rangle \leqslant \langle A(x,y), A(x,y)\rangle$ *for any* x *and* y, *then the system* (3.2.1) *has a non-trivial solution.*

This lemma was proved by Chern and Kuiper [48] for $n = 3$ and by Otsuki [127] for arbitrary n.

Proof of Theorem 3.2.1. Let $z_r = A^r_{ij} t^i t^j$ be an n-dimensional surface F of non-positive curvature in E^{2n-2}. We claim that at the point $t_i = 0$ there is a two-dimensional plane such that the curvature in the two-dimensional direction is zero. Let $x = \sum x^i e_i$ and $y = \sum y^i e_i$ be mutually perpendicular unit vectors in the tangent space of a point O of the surface F, let e_i be the unit vectors of the directions of the axes t^i, and let n_α be an orthonormal basis of normals. Then the curvature $K(x,y)$ of the surface F in the direction of the two-dimensional plane spanned by x and y is calculated by the formula

$$K(x,y) = [A(x,x)A(y,y) - A^2(x,y)]. \tag{3.2.2}$$

Let $x_0 = \sum x^i_0 e_i$ be a non-trivial unit vector solution of the system $A^r_{ij} x^i x^j = 0$ (it exists because the system satisfies the assumptions of the lemma). Since $A(x_0, x_0) = 0$, it follows from (3.2.2) that

$$K(x_0, y) = -A^2(x_0, y). \tag{3.2.3}$$

Let us consider the vector $A(x_0, y)$. Its components are $A^r_{ij} x^i_0 y^j = 0$. We take the direction of the vector x_0 as the direction of the axis t^n. Since $(x_0, y) = 0$, we obtain

$$A^r_{ij} x^i_0 y^j = A^r_{ns} y^s.$$

We introduce the vector A_{ns} with components A^r_{ns}; we can write $A(x_0, y) = A_{ns} y^s$. The vectors A_{ns} are of dimension $n-2$, and the number of these vectors is $n-1$, and hence the vectors A_{ns} are linearly dependent. Therefore, one can find y_0^s such that $\sum y_0^s = 1$ and $A_{ns}y_0^s = A(x_0, y_0) = 0$.

It follows from the formula (3.2.3) that the curvature of the surface F in the direction of the two-dimensional plane spanned by the vectors x_0 and y_0 vanishes, which proves the theorem.

Using this result, we can clarify the problem of non-immersibility of a Riemannian space R^n into a Riemannian space R^{2n-2} under some restrictions on the curvature of the spaces R^n and R^{2n-2}.

Let R^{2n-2} be a Riemannian space whose curvature in any two-dimensional direction is not less than c_0, where c_0 is an constant; we denote this space by $R^{2n-2}_{c_0}$. Let R^n be a Riemannian space whose curvature in the direction of any two-dimensional plane is not greater than $c_0 - c^2$, where $c > 0$ (we denote this space by $R^n_{c_0-c^2}$), let P be an arbitrary point of $R^n_{c_0-c^2}$, and let $Q(P)$ be an arbitrary neighbourhood of P on $R^n_{c_0-c^2}$.

The next assertion is a corollary to Theorem 3.2.1.

Theorem 3.2.2 [28]. *The neighbourhood $Q(P)$ of the point P of the space $R^n_{c_0-c^2}$ cannot be isometrically embedded in $R^{2n-2}_{c_0}$.*

Proof. Let $z_\rho = z_\rho(x)$ be an n-dimensional surface $R^n_{c_0-c^2}$ in $R^{2n-2}_{c_0}$. Then

$$R_{ijkl} = (A_{rik}A_{rjl} - A_{ril}A_{rjk}) + \overline{R}_{\alpha\beta\gamma\delta} z_i^\alpha z_j^\beta z_k^\gamma z_l^\delta, \tag{3.2.4}$$

where R_{ijkl} is the curvature tensor of $R^n_{c_0-c^2}$, $\overline{R}_{\alpha\beta\gamma\delta}$ is the curvature tensor of the enveloping space, A_{rij} are the components of the second fundamental forms for $n-2$ mutually perpendicular unit normals, and $z_i^\alpha = \dfrac{\partial z_\alpha}{\partial x_i}$.

In a neighbourhood of $R^n_{c_0-c^2}$ one can introduce coordinates such that the surface is defined by the equations

$$z_{n+r} = 0, \qquad z_i = x_i.$$

Then the equations (3.2.4) become

$$R_{ijkl} = \sum_r (A^r_{ik}A^r_{jl} - A^r_{il}A^r_{jk}) + \overline{R}_{ijkl}. \tag{3.2.5}$$

Let us calculate the curvature $R^n_{c_0-c^2}$ at the point $z_\rho = 0$. Let x and y be mutually perpendicular unit vectors belonging to the tangent space of $R^n_{c_0-c^2}$ at the point O. We denote by $K(x,y)$ and $\overline{K}(x,y)$ the curvatures of $R^n_{c_0-c^2}$ and $R^{2n-2}_{c_0}$, respectively, in the direction of two-dimensional plane spanned by the vectors x and y. It follows from (3.2.5) that

$$K(x,y) = \langle A(x,x), A(y,y)\rangle - \langle A(x,y), A(x,y)\rangle + \overline{K}(x,y). \tag{3.2.6}$$

By the assumption of the theorem, for mutually perpendicular unit vectors one has $K(x,y) \leqslant c_0 - c^2$ and $\overline{K}(x,y) \geqslant c_0$, and we see from (3.2.6) that

$$\langle A(x,x), A(y,y)\rangle - \langle A(x,y), A(x,y)\rangle \leqslant -c^2. \tag{3.2.7}$$

However, the coordinates x^i are chosen in such a way that the components of the metric tensor of the space $R^n_{c_0-c^2}$ at the point O are $g_{ij}=\delta_{ij}$; therefore,

$$\sum x_i^2 = 1, \qquad \sum y_i^2 = 1. \tag{3.2.8}$$

We define a surface in E^{2n-2} by the equation $z^r = A^r_{ij}x_ix_j$. It follows from (3.2.7) and (3.2.8) that this surface has strictly negative curvature at the point O; this contradicts Theorem 3.2.1, and thus proves Theorem 3.2.2.

Let us study the local properties of an isometric embedding of L^l in E^{2l-1}.

Let V be an n-dimensional vector space over the field of real numbers, and let $\Phi^1, \Phi^2, \dots, \Phi^n$ be symmetric bilinear forms on V. We say that $\Phi^1,\dots,\Phi^n$ are *exteriorly orthogonal* if

$$\sum_{\lambda=1}^{n}[\Phi^\lambda(x,y)\Phi^\lambda(z,w) - \Phi^\lambda(x,w)\Phi^\lambda(z,y)] = 0$$

for $x,y,z,w \in V$.

Theorem 3.2.3 (Cartan [44]). *Suppose that $\Phi^1,\dots,\Phi^n$ are exteriorly orthogonal symmetric bilinear forms on a real vector space V. Let the following condition hold: if a vector $X \in V$ is such that $\Phi^\lambda(x,y)=0$ for $1 \leqslant \lambda \leqslant n$ and for any $y \in V$, then $X = 0$. Then there exist a real orthogonal matrix (a^λ_μ) and n linear functionals $\varphi^1,\dots,\varphi^n$ such that*

$$\Phi^\lambda = \sum_\mu a^\lambda_\mu\, \varphi^\mu \otimes \varphi^\mu, \qquad 1 \leqslant \lambda \leqslant n.$$

This means that $\Phi^1,\dots,\Phi^n$ are simultaneously diagonalizable with respect to a basis dual to $\varphi^1,\dots,\varphi^n$.

Let M be an l-dimensional Riemannian manifold of constant sectional curvature k, let M be isometrically embedded in a Riemannian manifold of constant curvature K, and let $k < K$. If one considers the second fundamental forms $A^\lambda(x,y)$ with respect to an orthonormal basis of normals together with the form $\Psi = \sqrt{K-k}\,ds^2$, where ds^2 is the first fundamental form, then it follows from the Gauss formula that the forms $A^1(x,y),\dots,A^{l-1}(x,y),\Psi(x,y)$ are exteriorly orthogonal. In this case it follows from Theorem 3.2.3 that in any tangent space T_QM there is an orthonormal basis $e_1,\dots,e_l$ in which the first and second fundamental forms can be simultaneously reduced to diagonal form. These directions are called the *principal curvature directions* or simply *principal directions.* The curves tangent to the principal directions are called *principal curves.* A vector $X \in T_QM$ is said to be *asymptotic* if $A^\lambda(x,x)=0$ simultaneously for all second fundamental forms. It follows from the Bezout theorem and from Lemma 3.2.1 that there are 2^l asymptotic directions, which are of the form

$$\pm x^1 e_1 \pm x^l e_l,$$

that is, the asymptotic vectors are directed along the diagonals of the cube spanned by the principal directions.

Moore showed that the asymptotic curves can be chosen as coordinate curves. In the asymptotic coordinates, the line element becomes

$$ds^2 = \sum_{i=1}^{l} du_i^2 + 2\sum_{i\neq j} \cos\omega_{ij}\, du_i\, du_j,$$

that is, the coordinate net has the Chebyshev property [117]. Aminov showed that there is a system of coordinates in which the coordinate curves are the principal curves (see [7], [9], [10]), and the line element for the isometric immersion of the Lobachevsky space L^l into the Euclidean space E^{2l-1} can be represented as

$$ds^2 = \sum_{i=1}^{n} \sin^2\sigma_i\, du_i^2,$$

where $\sum_{i=1}^{n} \sin^2\sigma_i = 1$.

We note that systems of orthogonal coordinates in n-dimensional Euclidean space with the condition that the sum of the metric coefficients is constant were first considered by Guichard, and later by Darboux and Bianchi. In [7] it was proved that the integrability condition for the Codazzi–Ricci equations is given by the Gauss equations. Therefore, the entire system of equations for an immersion of L^n into E^{2n-1} reduces to the system

$$R_{ijij} = \sin^2\sigma_i \sin^2\sigma_j, \qquad R_{ijkj} = 0, \quad i\neq k, \qquad \sum_{i=1}^{n}\sin^2\sigma_i = 1. \tag{3.2.9}$$

It was proved that, to construct an arbitrary local immersion of L^n into E^{2n-1}, it is necessary and sufficient to indicate a way of constructing a system of orthogonal coordinates in L^n that satisfies the condition $\sum_{i=1}^{n}\sin^2\sigma_i = 1$, which leads to the system (3.2.9). The system (3.2.9) is a multidimensional analogue of the sine-Gordon equation $\omega_{xx} - \omega_{yy} = \sin\omega$ or, in another coordinate system, the equation $\omega_{xy} = \sin\omega$.

Cartan showed that an analytic isometric embedding of the Lobachevsky space L^n in E^{2n-1} depends on $n(n-1)$ arbitrary functions of one variable [45], [78].

Properties of an isometric immersion of the Lobachevsky space L^l into E^{2l-1} were studied in a cycle of papers of Aminov. In particular, if an immersion of L^3 into E^5 satisfies a certain geometric condition, then the main system of immersion contains as a subsystem the equations of motion of a rigid body about its centre of mass in the Newton gravitational field [11]. The system of differential equations for an isometric immersion of L^4 into E^7 can be associated with the system of Maxwell equations. The electromagnetic stress tensor is introduced in the natural way.

The Grassmann image of L^l in E^n has also been investigated. If to any point $P\in L^l \subset E^n$ one assigns the normal space to L^l at this point subjected to the parallel displacement to a chosen point of E^n, then we obtain the Grassmann map of L^l into the Grassmann manifold $G(n-l,n)$ of $(n-l)$-dimensional planes, and we denote the image of this map by Γ. The metric of the Grassmann image Γ_2 of the Lobachevsky space L^l in E^{2l-1} is of the form

$$d\sigma^2 = \sum_{i=1}^{n} \cos^2\sigma_i\, du^2.$$

For the Lobachevsky space L^l in E^{2l-1}, the sectional curvature of the Grassmann image $G(l-1, 2l-1)$ by two-dimensional planes tangent to the Grassmann image Γ belongs to the interval $(0, 1)$ [7]. We note that the curvature of the Grassmann manifold varies within the interval $[0, 2]$.

Theorem 3.2.4 (Aminov [12]). *If ds^2 is the metric of the Lobachevsky space L^l and $d\sigma^2$ is the metric of the Grassmann image of an isometric immersion of a domain in L^l into E^n with flat normal connection, then the metric*

$$ds^2 + d\sigma^2$$

is flat.

Theorem 3.2.5 (Aminov [13]). *There is no local isometric immersion of L^3 into E^5 of class C^3 for which the metric of the Grassmann image is of constant sectional curvature.*

Many authors have tried to generalize the Hilbert theorem to the multidimensional case. Results in this direction have been obtained under diverse additional conditions.

Theorem 3.2.6 (Aminov [9]). *There is no immersion of the entire Lobachevsky space for which the principal curves of one of the families are geodesics.*

Theorem 3.2.7 (Aminov [11]). *There is no regular isometric immersion of the complete Lobachevsky space L^3 into E^5 of class C^3 with a family of totally geodesic curvature surfaces.*

In [117] Moore proved that there is no isometric immersion of the complete Lobachevsky space L^l into E^{2l-1} in the form of a minimal submanifold. In [115] this fact was generalized to the isometric immersions of L^l into $E^{l+\rho}$ $(\rho \geqslant l-1)$ with flat normal connection. We note that an isometric immersion of L^l into E^{2l-1} (for $\rho = l-1$) has a flat normal connection.

If one is isometrically immersing a hyperbolic manifold (a complete Riemannian manifold that is not simply connected and has constant negative sectional curvature) rather than the complete simply connected Lobachevsky space, then there are best possible results in a sense: in [176] Xavier proved that an isometric immersion of a hyperbolic manifold H^l into E^{2l-1} is impossible if H^l has a non-elementary fundamental group (that is, the fundamental group has an infinite limit set on the absolute). This condition is equivalent to the property that the fundamental group has at least two generators.

This result was generalized in [128] to the case in which H^l is a Fuchsian manifold. The most advanced form of the result is as follows.

Theorem 3.2.8 (Nikolayevsky [124]). *A complete not simply connected hyperbolic manifold H^l admits no regular isometric immersion into E^n with flat normal connection.*

Corollary. *A complete not simply connected hyperbolic manifold H^l admits no regular isometric immersion into E^{2n-1}.*

This follows from the fact that an isometric immersion of H^l into E^{2l-1} has a flat normal connection.

Proof of Theorem 3.2.8. Let $f\colon H^l \to E^n$ be an isometric immersion of H^l with flat normal connection. It follows from the Ricci equations that at any point of $F^l = f(H^l)$ there are principal directions. By [7] these directions are holonomic, and hence the principal curves can be chosen as coordinate curves. The local metric g on F^l becomes

$$ds^2 = \sum_{i=1}^{l} \sin^2 \sigma_i \, (du_i)^2$$

in these coordinates, and the metric of the Grassmann image F^l is

$$d\sigma^2 = \sum \cos^2 \sigma_i \, du_i^2.$$

We introduce a flat metric g_0 on H^l,

$$ds_0^2 = \sum_{i=1}^{l} (du_i)^2,$$

which is locally flat. Let us consider a universal covering $\widetilde{H}^l$ of the manifold H^l; $\widetilde{H}^l$ is homeomorphic to the Euclidean space E^n.

We lift the metrics g and g_0 to E^n. The metric g_0 is also complete. This follows from the completeness of the metric g and from the fact that the lengths of the corresponding curves in the metric g_0 are greater than those in the metric g. The space E^n is globally isometric to the Euclidean space with respect to the lifted metric g_0. Thus, we have two metrics on E^n: the hyperbolic metric g and the Euclidean metric g_0. The action of the fundamental group $\pi_1(H^l)$ preserves the first and second fundamental forms of $F^l = f(H^l)$, and hence the metric of the Grassmann image as well. Therefore, a non-trivial element of $\pi_1(H^l)$ is an isometry on E^n with respect to each of the metrics g and g_0 simultaneously and sends the coordinate curves u_i to themselves. Let γ be a non-trivial element of $\pi_1(H^l)$, and let $u(u_1, \ldots, u_l)$ be the coordinates. Then $\gamma(u) = Au + b$, where A is an orthogonal matrix that consists of $0, \pm 1$ only.

There is an m such that A^m is the identity matrix. Then $\Phi(u) = \gamma^m(u) = u + c$, where $c(c^1, \ldots, c^l) = \text{const}$, $c \neq 0$, because otherwise γ has a fixed point and $H^l = E^l/\pi_1(H^l)$ is not a manifold. The isometry $\Phi = \gamma^m$ of the metric g_0 is a parallel translation, and $\text{dist}_0(u, \Phi(u)) = \|c\| = \text{const}$. On the other hand, the isometry Φ of the metric $g < g_0$ satisfies the condition $\text{dist}(u, \Phi(u)) < \text{dist}_0(u, \Phi(u)) = \text{const}$. However, as is well known, an isometry of a Lobachevsky space with this property is the identity transformation. We arrive at a contradiction.

For other space forms used as enveloping spaces one has the following assertion.

Theorem 3.2.9 [31]. *Let F^l be a compact l-dimensional surface of constant sectional curvature c in a space of constant curvature C. If $c < C$, then $C > 0$, $c = 0$, and the surface is locally isometric to the Euclidean space.*

Proof. In [117] it was proved that the case $0 < c < C$ is impossible. The surface F^l admits a field of principal directions, and this field is holonomic. There is a compact parallelizable covering manifold M^l of finite multiplicity. The parallelization of M^l is formed by the vector fields of principal directions. In a neighbourhood of any point of the manifold M^l one can take the integral trajectories of the vector fields as the coordinate curves of the parallelization (these trajectories parallelize the manifold because the vector fields are holonomic). Hence, one can introduce a locally Euclidean metric on M^l. The covering space of a compact locally Euclidean manifold is the torus T^l. It must be a covering space for the submanifold F^l as well. In this case a metric of constant curvature C is induced on the torus. However, as is known, if the torus T^l is endowed with a Riemannian metric with sectional curvature of constant sign, then this metric is flat [108]. Hence, $c = 0$. By the assumption of the theorem one has $C > 0$.

The field of asymptotic directions was used in [31].

Isometric embeddings of the Lobachevsky space L^l, $l \geqslant 3$, in a Hilbert space were constructed by Blanuša [18]. The first embeddings in a finite-dimensional space E^n were constructed in [23] for $l \geqslant 3$ and $n = 6l - 5$ and for $l = 2$ and $n = 6$; these C^∞ embeddings were given by explicit formulae. The Nash and Gromov theorems give a higher estimate for n.

In [87] Henke presented C^∞ isometric immersions of the Lobachevsky space $L^l(C)$ of curvature C into a spherical space of curvature $\widetilde{C}$ for $C < 0 < \widetilde{C}$, and immersions $L^l(C) \to L^{4n-3}(\widetilde{C})$ for $C < \widetilde{C} < 0$.

Let $M^l(\widetilde{M}^n)$ be space forms of constant curvature, and let $n \geqslant 4l - 3$. If M^l is simply connected, then there is a C^∞ isometric immersion of M^l into M^n. An isometric immersion of the Lobachevsky space L^l into E^{4l-3} was given in [86].

In [14], Azov considered two classes of Riemannian metrics,

$$ds^2 = du_1^2 + f(u_1) \sum_{i=2}^{l} du_i, \quad f > 0, \tag{3.2.10}$$

$$ds^2 = g^2(u_1) \sum_{i=2}^{l} du_i^2, \quad g > 0, \tag{3.2.11}$$

and solved the problem of their immersibility into E^n and S^n, $n > l$. We assume that the functions f and g are of class C^1. In the special case $g(u_1) = \frac{1}{u_1}$, $u_1 > 0$, we obtain the metric of the Lobachevsky space. The following theorems were proved.

Theorem 3.2.10 (Azov [14]). *The metrics* (3.2.10) *and* (3.2.11) *admit isometric immersions into the Euclidean space* E^{4l-3}.

Theorem 3.2.11 (Azov [14]). *The metrics* (3.2.10) *and* (3.2.11) *admit immersions into the spherical space* S^{4l-3}.

The proofs of these assertions were carried out by methods suggested by Blanuša, and the specific parametric equations of the surfaces were given. The smoothness class of these surfaces coincides with that of the functions f and g.

Following Blanuša [23], we introduce some auxiliary functions. Let

$$A=\int_0^1 \sin\pi\xi e^{-\frac{1}{\sin^2\pi\xi}}\,d\xi,$$

$$\varphi_1(u)=\left(\frac{1}{A}\int_0^{u+1}\sin\pi\xi e^{-\frac{1}{\sin^2\pi\xi}}\,d\xi\right)^{1/2},\quad \varphi_2(u)=\left(\frac{1}{A}\int_0^{u}\sin\pi\xi e^{-\frac{1}{\sin^2\pi\xi}}\,d\xi\right)^{1/2}.$$

We introduce the following notation: $a=\max(|\varphi_1'|,|\varphi_2'|)$,

$$\psi_{1,f}(u)=2(a+1)\Big\{\Big[\max_{2k-1\leqslant u\leqslant 2k+1}(|f|,|f'|)\Big]+1\Big\},\quad 2k-1\leqslant u<2k+1,$$
$$\psi_{2,f}(u)=2(a+1)\Big\{\Big[\max_{2k-2\leqslant u\leqslant 2k}(|f|,|f'|)\Big]+1\Big\},\qquad 2k-2\leqslant u<2k,$$

where the square brackets stand for the integral part of the bracketed expression. Let

$$h_{1,f}=\frac{f(u)\,\varphi_1(u)}{\psi_{1,f}(u)},\quad h_{2,f}=\frac{f(u)\,\varphi_2(u)}{\psi_{2,f}(u)},$$
$$\alpha_f(u,t)=\sqrt{l-1}\cdot t\,\psi_{1,f}(u),\quad \beta_f(u,t)=\sqrt{l-1}\cdot t\,\psi_{2,f}(u).$$

An immersion of the metric (3.2.10) into the Euclidean space E^{4l-3} can be given by the following parametric equations:

$$x_0=\int_0^{u_1}\sqrt{1-h_{1,f}'^2(\xi)-h_{2,f}'^2(\xi)}\,d\xi,$$
$$x_{i1}=\frac{h_{1,f}(u_1)}{\sqrt{l-1}}\cos\alpha_f(u_1,u_i),\quad x_{i2}=\frac{h_{1,f}(u_1)}{\sqrt{l-1}}\sin\alpha_f(u_1,u_i),$$
$$x_{i3}=\frac{h_{2,f}(u_1)}{\sqrt{l-1}}\cos\beta_f(u_1,u_i),\quad x_{i4}=\frac{h_{2,f}(u_1)}{\sqrt{l-1}}\sin\beta_f(u_1,u_i),\quad i=2,\dots,l. \tag{3.2.12}$$

Let us introduce the function

$$\Phi(u)=\int_0^u\frac{[(1-h_{1,f}'^2-h_{2,f}'^2)(1-h_{1,f}^2-h_{2,f}^2)-(h_{1,f}'\,h_{1,f}+h_{2,f}'\,h_{2,f})^2]^{1/2}}{1-h_{1,f}^2-h_{2,f}^2}\,d\xi,$$

where $h_{j,f}=h_{j,f}(\xi)$ for $j=1,2$. An immersion of the metric (3.2.10) in the spherical space S^{4l-3} can be given by the following equations:

$$x_{01}=\sqrt{1-h_{1,f}^2(u_1)-h_{2,f}^2(u_1)}\cdot\cos\Phi(u_1),$$
$$x_{02}=\sqrt{1-h_{1,f}^2(u_1)-h_{2,f}^2(u_1)}\cdot\sin\Phi(u_1),$$
$$x_{i1}=\frac{h_{1,f}(u_1)}{\sqrt{l-1}}\cos\alpha_f(u_1,u_i),\quad x_{i2}=\frac{h_{1,f}(u_1)}{\sqrt{l-1}}\sin\alpha_f(u_1,u_i),$$
$$x_{i3}=\frac{h_{1,f}(u_1)}{\sqrt{l-1}}\cos\beta_f(u_1,u_i),\quad x_{i4}=\frac{h_{1,f}(u_1)}{\sqrt{l-1}}\sin\beta_f(u_1,u_i),\quad i=2,\dots,l. \tag{3.2.13}$$

The proofs of Theorems 3.2.10 and 3.2.11 for the metric (3.2.11) can be carried out in a quite similar way. One need only modify the step functions ψ_1 and ψ_2 as follows:

$$\psi_{1,g} = 2(a+1)\left\{\left[\max_{2k-1\leqslant u\leqslant 2k+1}\left(1, \left|\frac{g'}{g}\right|\right)\right] + 1\right\}, \quad 2k-1 \leqslant u \leqslant 2k+1,$$
$$\psi_{2,g} = 2(a+1)\left\{\left[\max_{2k-2\leqslant u\leqslant 2k}\left(1, \left|\frac{g'}{g}\right|\right)\right] + 1\right\}, \quad 2k-2 \leqslant u \leqslant 2k.$$

Moreover, one must replace the function $1-h_{1,f}'^2-h_{2,f}'^2$ by the function $g^2-h_{1,g}'^2-h_{2,g}'^2$.

3.3. Isometric immersions in the form of submanifolds of positive extrinsic curvature.

3.3.1. *k*-umbilical submanifolds. As is known, a regular hypersurface $F^l(c)$, $l \geqslant 3$, of constant sectional curvature c in the space $M^{l+1}(\widetilde{c})$ of constant sectional curvature has locally the following properties:

1) either $c = \widetilde{c}$ and the external null-index μ satisfies the condition $\mu \geqslant l-1$;
2) or $c > \widetilde{c}$ and the hypersurface is umbilical, that is, all principal curvatures are equal and constant [58], [159].

The umbilical hypersurfaces are spheres in the spherical and Euclidean spaces, and spheres, equidistant hypersurfaces, and horospheres in the Lobachevsky space. Under an isometric immersion of a space $M^l(c)$ of constant curvature c in a space $M^{l+p}(\widetilde{c})$ of constant curvature, where $c > \widetilde{c}$, some umbilical properties of the hypersurfaces are inherited for sufficiently small codimension $p < l$. A point P of a submanifold $M^l(c) \subset M^{l+p}(\widetilde{c})$, $c > \widetilde{c}$, is said to be *isotropic* if the normal curvature vectors $A(x,x)$ are the same in all directions.

O'Neill [126] proved that for $p \leqslant l-2$ any tangent space $T_Q M^l(c)$, $Q \in M^l(\widetilde{c})$, contains an isotropic subspace $L(Q)$ of dimension $\rho(Q) \geqslant l-p+1$ (that is, the normal curvature vectors in the directions belonging to this subspace are the same). If the distribution $L(Q)$ and the fields of normal curvature vectors are regular, then the distribution $L(Q)$ is integrable, and each leaf is an isotropic submanifold of $M^{l+p}(\widetilde{c})$ [126].

For submanifolds of curvature $c > 0$ one has the following global result.

Theorem 3.3.1 (O'Neill [126]). *Let $M^l(c)$, $c > 0$, be a complete regular submanifold in $M^{l+2}(\widetilde{c})$, where $l \geqslant 4$ and $c > \widetilde{c}$. Then $M^l(c)$ contains an isotropic point.*

The theorem is false if $c \leqslant 0$ or $p > 2$. For $c < \widetilde{c}$ and for codimension $p = l-1$ it follows from Theorem 3.2.3 that the second fundamental forms of the submanifold are orthogonally diagonalizable, that is, the first and second fundamental forms can simultaneously be reduced to diagonal forms. For $c > \widetilde{c}$ one has a dual result.

Lemma 3.3.1 (Moore [118]). *Let $c - \widetilde{c} = k > 0$ and $p \leqslant l-1$. If at a point $P \in M^l(c) \subset M^{l+p}(\widetilde{c})$ there is a unit vector $x \in T_Q M^l$ such that the squared length $\langle A(x,x), A(x,x)\rangle$ of the normal curvature vector is less than k, then $p = l-1$, and the second fundamental form of $M^l(c)$ is orthogonally diagonalizable.*

Definition. A point $P \in M^l(c) \subset M^{l+p}(\widetilde{c})$, $c - \widetilde{c} = k > 0$, is said to be *strongly umbilic* if there is a unit normal vector $n \in N_P M^l$ such that the second fundamental form can be represented as

$$A(x,y) = \sqrt{k}\ \langle x, y\rangle\, n \tag{3.3.1}$$

for $x, y \in T_P M^l$.

The submanifolds of strongly umbilic points in the spherical and Euclidean spaces are spheres in some totally geodesic space $M^{l+1}(\widetilde{c})$; in the Lobachevsky space these are spheres, equidistant surfaces, and horospheres in some totally geodesic space. If $M^l(c)$ is a complete regular submanifold of $M^{l+2}(\widetilde{c})$, where $c > 0$ and $c - \widetilde{c} > 0$, then $M^l(c)$ has a strongly umbilic point [126]. The result fails for $c < 0$ and for $p > 2$.

Definition. A point $P \in M^l(c) \subset M^{l+p}(\widetilde{c})$, $c - \widetilde{c} = k > 0$, is said to be *weakly umbilic* if there is a unit normal vector $n \in N_P M^l$ such that the second fundamental form A satisfies the equation

$$\langle A(x,y), n\rangle = \sqrt{k}\ \langle x, y\rangle \tag{3.3.2}$$

for $x, y \in T_P M$.

Lemma 3.3.2 (Moore [118]). *Let $c - \widetilde{c} = k > 0$ and $p \leqslant l - 1$. If the squared lengths $\langle A(x,x), A(x,x)\rangle$ of the normal curvature vectors are not less than k for any $x \in T_P M^l$, then the point $P \in M^l$ is weakly umbilic.*

It follows from Lemmas 3.3.1 and 3.3.2 that for $p \leqslant l-2$ any point of M^l is weakly umbilic; for $p = l - 1$ any point either is weakly umbilic or the second fundamental forms can be orthogonally diagonalized. However, the following assertion holds in the global case.

Theorem 3.3.2 (Moore [118]). *Let $c - \widetilde{c} > 0$ and $c > 0$. If $M^l(c) \subset M^{2l-1}(\widetilde{c})$ is a complete regular submanifold of constant positive curvature, then any point of $M^l(c)$ is weakly umbilic.*

For a submanifold F^l of a space of constant curvature $M^{l+p}(\widetilde{c})$ one can define the index of 'strong umbilicity' $s(P)$. At a point $P \in F^l$ this index is the dimension of a maximal subspace $L(P) \subset T_P F^l$ such that the restriction of the second fundamental form of F^l to $L(P)$ can be represented as

$$A(x,y) = \sqrt{k}\ \langle x, y\rangle\ n \tag{3.3.3}$$

for any $x \in L(P)$ and $y \in T_P F^l$.

If F^l is isometric to $M^l(c)$, $k = c - \widetilde{c} > 0$, then the index of strong umbilicity satisfies the inequality $s(Q) \geqslant l - p + 1$, $Q \in M^l(c)$, for $p \leqslant l - 2$, and also for $p = l - 1$ if the point is weakly umbilic. Indeed, if at the point $Q \in M^l(c)$ we take an orthonormal basis of normals $n_1, \dots, n_{p-1}, n_p = n$, where n is a unit normal vector mentioned in Lemma 3.3.2, then it follows from the Gauss formula that the second fundamental forms with respect to the normals n_i $(i = 1, \dots, p-1)$ satisfy the identity

$$\sum_{\alpha=1}^{p-1} A^\alpha(x,x)A^\alpha(y,y) - A^\alpha(x,y)A^\alpha(x,y) = 0,$$

that is, the restriction of the external null-index of $M^l(c)$ to the subspace of normals orthogonal to n satisfies the inequality $\mu(Q) \geqslant l - p + 1$. We recall that $\mu(Q)$ is the dimension of the maximal subspace $L(Q) \subset T_Q M^l$ such that $A(x,y) = 0$ for $x \in L(Q)$, $y \in T_Q M^l$. The restriction of the second fundamental forms of $M^l(Q)$ to $L(Q)$ satisfies (3.3.3), and for the index of strong umbilicity one has $s(Q) = \mu(Q) \geqslant l - p + 1$. Using the properties of the isotropicity index [120], one can readily show that for $M^l(c) \subset M^{l+p}(\widetilde{c})$, $c - \widetilde{c} > 0$, one has $\rho(Q) = s(Q)$, and the maximal isotropic space coincides with the maximal strongly umbilical space. We note that strong k-umbilicity is a special case of strong k-convexity [37].

If the distribution $L(Q)$ is regular and the normal vector field $n(Q)$ is regular and parallel in the normal bundle, then the distribution $L(Q)$ is involutive, and the leaves are umbilical surfaces both in $M^l(c)$ and in the enveloping space $M^{l+p}(\widetilde{c})$. If the index of strong umbilicity is minimal at a point P, then the distribution $L(Q)$ is regular in a neighbourhood of P. If the dimension of the normal space is constant, then the normal vector field n is regular. The condition that a surface be strongly umbilical implies the following consequence.

Theorem 3.3.3 (Moore [118]). *Let M^l be a compact Riemannian manifold of constant sectional curvature $c > 0$. Let M^l be isometrically and regularly immersed into the Euclidean space E^{l+p}, where $p \leqslant l/2$. Then M^l is simply connected and isometric to a standard sphere.*

The Euclidean space can be replaced here by a simply connected space $M^{l+p}(\widetilde{c})$ with $c > \widetilde{c}$. Partial results for the codimension two case are obtained in [85].

3.3.2. Extension of isometric immersions. Isometric immersions of an l-dimensional sphere into the Euclidean space E^{l+2} were studied in the papers of Henke [84], [85] and Erbacher [66]. One can readily construct such immersions by considering an immersion of E^{l+1} (or of a spatial neighbourhood of S^l) into E^{l+2} (for instance, by winding around a cylinder or a cone) and by restricting this map to S^l. There is a question as to whether an arbitrary isometric C^∞ immersion $f\colon S^l \to E^{l+2}$ can be regarded as a restriction of an isometric C^∞ immersion of E^{l+1} into E^{l+2}. If a map f is a restriction to S^l of some isometric map $F\colon E^{l+1} \to E^{l+2}$, then f is said to be *extendable*, and the map F is called an *isometric extension of* f. In other words, this means that a local (or global) isometric immersion f of the standard sphere S^l into E^{l+2} is a composition $f = h \circ g$ of the standard embedding $g\colon S^l \to E^{l+1}$ and an isometric immersion $h\colon E^{l+1} \to E^{l+2}$. It turns out that the existence of such an extension to a neighbourhood of a point $x \in S^l$ depends on whether or not the point x is a boundary point of the set of umbilic points of the immersion f.

Similar problems for an isometric immersion of E^2 into E^4 were considered in Chapter 2. In [66] Erbacher proved the existence of an isometric extension F of a map $f\colon S^l \to E^{l+2}$, $l \geqslant 4$, in a neighbourhood of a non-umbilic point $x \in S^l$.

In [85] Henke proved the uniqueness of such an extension if x is a non-umbilic point, that is, for these points one has a certain kind of 'rigidity'. If x is an interior point of the set of umbilic points, then a local isometric extension of f in a neighbourhood of x also exists, but it is not unique; on the contrary, there are two local isometric extensions of f that are defined in a neighbourhood and are distinct in any neighbourhood. It follows from these assertions that if $l \geqslant 4$, then a local isometric immersion $S^l \to E^{l+2}$ has an extension at any point of S^l except for the boundary set of the set of all umbilic points. In [85] an example of an isometric C^∞ immersion $f\colon S^l \to E^{l+1}$ was constructed for which there is a point $p \in S^l$ such that f is not locally isometrically extendable to any neighbourhood of p. The geometric structure of the set Z of non-umbilic points has also been studied. There are two affine hyperplanes, $H_1, H_2 \subset E^{l+1}$, such that $H_1 \cap S^l \neq \varnothing$, $H_2 \cap S^l \neq \varnothing$, $H_1 \cap H_2 \cap S^l \neq \varnothing$, and the intersections of the open half-spaces H_1^+ and H_2^+ with S^l give Z, that is, $Z = H_1^+ \cap H_2^+ \cap S^l$.

The following theorem answers the question of global isometric extensions.

Theorem 3.3.4 (Moore [120]). *Let S^l be the standard sphere of unit curvature regarded as the boundary of the unit ball D^{l+1}. If $l \geqslant 3$ and if $f\colon S^l \to E^{l+2}$ is a C^∞ isometric immersion, then there is a unique C^∞ isometric immersion $\widetilde{f}\colon D^{l+1} \to E^{l+2}$ such that $\widetilde{f}/S^l = f$.*

The theorem fails if one replaces C^∞ by C^k. Let $\widetilde{f}\colon E^{n+1} \to E^{n+2}$ be given in the form $\widetilde{f}(x_1, \dots, x_{l+1}) = (x_1, \dots, x_{l+1}, |x_1 + 1|^\alpha)$, where $k < \alpha < k+1$; f is a cylindrical isometric immersion of class C^k, but not of class C^{k+1}. The composition of the standard embedding of the unit sphere in E^{l+1} centred at the origin is a C^{2k} isometric embedding, which can be uniquely extended to an isometric immersion of D^{l+1} into E^{l+2}, and this immersion is not C^{k+1} regular at the point $(-1, 0, \dots, 0)$. As a consequence we see that any C^∞ isometric immersion $S^l \to E^{l+2}$ is homotopy equivalent by a C^∞ isometric immersion to the standard embedding of the sphere in a hyperplane E^{l+1}. In particular, this implies that a smooth isometric immersion of the sphere S^l ($l \geqslant 3$) into E^{l+2} cannot be knotted. For $l \geqslant 4$ the above assertion on the homotopy equivalence was established in [174] without proving that the homotopy is differentiable. For finite-order smoothness, the assertion was proved in [119].

3.3.3. Isometric immersions of metrics of constant curvature by minimal submanifolds. Let $\phi\colon M^n \to E^n$ be an isometric immersion of M into a Euclidean space. Then the mean curvature vector satisfies the equation $\Delta\phi = nH$. If the coordinate functions of the immersion are eigenfunctions corresponding to the same eigenvalue λ, then

$$H = \lambda\phi/n.$$

Since $\langle H, d\phi\rangle = 0$, it follows that $\langle \phi, d\phi\rangle = 0$, $|\phi|^2$ is constant, and ϕ is an immersion into the sphere S^{N-1} of radius $\sqrt{n/\lambda}$. Moreover, it is a minimal immersion into the sphere.

The eigenfunctions on the standard sphere are the restrictions of the homogeneous harmonic polynomials in E^{n+1} to the sphere $S^n(1)$. Corresponding to any eigenvalue $\lambda_d = d(d+n+1)$ is an eigenfunction subspace E_{λ_d} of dimension

$$N_d = \frac{(2d+n-1)(d+n-2)}{d!\,(n-1)!}\,.$$

For d odd these functions define standard minimal isometric immersions $f_{n,d}$ of the standard unit sphere $S^n(1)$ into the sphere $S^{N_d-1}(\sqrt{n/\lambda_d}\,)$. For n even we obtain an isometric embedding of the real projective space $\mathbb{R}P^n$ in $S^{N_d-1}(\sqrt{n/\lambda_d}\,)$.

Theorem 3.3.5 (Calabi [42]). *Let S be a two-dimensional sphere with constant Gaussian curvature K, and let*

$$f\colon S^2 \to S^{n-1}(r) \subset E^n \quad (n = 2m+1 \geqslant 3)$$

be an isometric minimal immersion of S^2 such that the image does not belong to any hyperplane of E^n (the so-called complete immersion). Then:

1) *the Gaussian curvature K can take only one of the values*

$$K = \frac{2}{m(m+1)r^2}\,;$$

2) *the immersion is uniquely defined up to a motion in the sphere S^{n-1}, and the components of the radius vector of the isometric immersion are appropriately normalized spherical harmonics of order m on the sphere S^2 that form an orthonormal basis.*

Even locally, the sphere S^n admits no minimal two-dimensional surfaces of constant negative Gaussian curvature [39], [95].

All minimal two-dimensional surfaces of Gaussian curvature $K = 0$ in the spherical space were found in [102]. The only minimal surfaces of constant Gaussian curvature in a Euclidean space are domains on totally geodesic two-dimensional planes of zero Gaussian curvature [39]. A similar assertion holds for the Lobachevsky space H^n: a minimal surface of constant Gaussian curvature in the Lobachevsky space of curvature -1 can only be a domain on a totally geodesic Lobachevsky plane $H^2 \subset H^n$ of curvature -1 [103].

Do Carmo and Wallach [61] showed that for $d \leqslant 3$ any complete spherically minimal isometric immersion $f\colon S^m \to S^n$ of a sphere of constant sectional curvature is standard. It was also shown that for $n > 2$ there are many other spherically minimal isometric immersions of $S^n(1)$ into $S^N(r)$ that differ from standard ones. These immersions are parametrized by a convex body in a finite-dimensional vector space. The exact values of dimensions of these convex bodies were found in [166]; for new constructions of minimal spherical immersions, see [167].

The question about the least dimension N for which there is a minimal isometric immersion of $S^n(1)$ into $S^N(r)$ that is not totally geodesic was posed in [61]. A lower bound was given by Moore [117]. Do Carmo and Wallach conjectured [61] that $N = \frac{n(n+3)}{2} - 1$, and that the desired immersion is the Veronese embedding, that is, the standard embedding corresponding to the case $d = 2$. However, this is not the case, at least for $n = 3$. Ejiri showed that there is a minimal isometric immersion of $S^3(1)$ into $S^6(\frac{1}{4})$ that is not totally geodesic [63], [68].

In [168] it is proved that there is no isometric minimal immersion of the sphere S^m of curvature $\frac{m}{4m+3}$ into the standard unit sphere S^n for $n < \frac{(m+2)(m+7)}{12} - 1$.

The following assertion holds for homogeneous spherical forms.

Theorem 3.3.6 (Deturk and Ziller [59]). *Any homogeneous spherical form admits a spherically minimal isometric embedding in a standard sphere* (*of sufficiently high dimension and of the corresponding radius*).

We note that a homogeneous spherical form belongs to one of the following classes:

1) $M = S^3/\Gamma$, where Γ is a finite subgroup of $S^3 = SU(r) = S_p(1)$;

2) $M = S^{4n-1}/\Gamma$, where Γ is a finite subgroup of $S_p(1)$ that acts on $E^{4n} = H^n$ by multiplying each of the quaternion coordinates from the left;

3) $M = S^{2n-1}/\mathbb{C}_d$, where $\mathbb{C}_d$ is generated by $e^{2\pi i/d}$ and acts on $E^{2n} = \mathbb{C}^n$ by multiplying each of the complex coordinates.

The minimal spherical immersions of non-homogeneous spherical forms were investigated in [67].

3.3.4. Isometric immersions of projective spaces into Euclidean spaces. First of all, there are topological obstructions to isometric immersion of real projective spaces with standard metric of constant sectional curvature. We recall that an immersion of a differentiable manifold M^l into a differentiable manifold V^{l+p} is a map f of regularity class C^1 with non-zero Jacobian. If this map is a diffeomorphism onto a submanifold, then one speaks of an embedding.

Whitney proved that any manifold M^l can be immersed into E^{2l-1} and embedded in E^{2l}. If we restrict ourselves to the real projective spaces $\mathbb{R}P^l$, then $\mathbb{R}P^l$ can be immersed into E^{l+p} if and only if there is a field over $\mathbb{R}P^l$ formed by orthonormal l-frames tangent to $\mathbb{R}P^{l+p}$ [95].

Let $\alpha(l)$ be the number of 1s in the binary representation of l. If l is a power of two, then $\alpha(l) = 1$.

Theorem 3.3.7 [95]. a) *Let* $l = 2k + 1$ *and* $l > 1$. *Then* $\mathbb{R}P^l$ *can be embedded in* E^{2l-1}.

b) *If* $l > 1$, *then* $\mathbb{R}P^l$ *can be immersed into* E^{2l-1}, *and* $\mathbb{R}P^3$ *into* E^4.

Let $l > 3$. *If* $l = 1 \bmod 2$, *then* $\mathbb{R}P^l$ *can be immersed into* E^{2l-3}. *If* $l = 3 \bmod 4$, *then* $\mathbb{R}P^l$ *can be immersed into* E^{2l-5}.

Theorem 3.3.8 [95].

a) *Let* $l = 2r - 1$, *where* $r \geqslant 4$. *Then* $\mathbb{R}P^l$ *cannot be immersed into* E^{2n-q}, *where*

$$\begin{aligned} q &= 2r \quad & (r = 1, 2 \mod 4), \\ q &= 2r + 1 \quad & (r = 0 \mod 4), \\ q &= 2r + 2 \quad & (r = 3 \bmod 4). \end{aligned}$$

b) $\mathbb{R}P^l$ *can be immersed into* E^{2l-q+1}.

Theorem 3.3.9 [95]. *If* l *is odd and* $\alpha(l) > 3$, *then* $\mathbb{R}P^l$ *can be immersed into* E^{2l-8}. *If* $l = 3 \bmod 4$ *and* $\alpha(l) > 5$, *then* $\mathbb{R}P^l$ *can be immersed into* E^{2l-9}.

For other results in this direction, see [95] and [109].

When considering a projective space with the standard metric, some metric restrictions must be added. In particular, the following assertion holds.

Theorem 3.3.10 (Gromov and Rokhlin [80]). *If l is a power of two and if $l > 1$, then the l-dimensional real projective space endowed with a Riemannian metric having positive scalar curvature everywhere, in particular, an l-dimensional elliptic space, admits no isometric C^2 embedding in E^{2l}.*

The proof is somewhat more sophisticated than in the two-dimensional case. Instead of the Whitney theorem, the following generalization due to Mahowald is used: a real projective space of dimension l admits no differentiable embedding in E^{2l} with normal vector field without singularities for the above values of l. By virtue of Mahowald's theorem, it suffices to prove that if V is a Riemannian manifold of arbitrary dimension $l \geqslant 2$ and with positive scalar curvature everywhere, then for any isometric C^2 immersion $f\colon V \to E^q$ with $q > l$ there is a normal vector field on V without singularities.

Such a field can be constructed for any $l \geqslant 2$ and $q > l$ in just the same way as done above for $l = 2$ and $q = 4$ as a field dual to the trace σ of the second fundamental form. As above, the only fact that needs a proof is that the linear form defined by the trace on the normal plane N_v is not identically zero. Let $e_1, \dots, e_l$ and $\xi_1, \dots, \xi_l$ be orthonormal bases in the tangent plane T_v and in the normal plane N_v, respectively, and let $(a^k_{ij})^n_{i,j=1}$ be the matrices of the fundamental form with respect to the basis $e_1, \dots, e_l$. According to the Gauss formula, the curvature of the manifold V at the point v in the two-dimensional direction determined by the vectors e_i and e_j is equal to

$$\sum_k [a^k_{ii} a^k_{jj} - (a^k_{ij})^2],$$

and hence the scalar curvature at this point is given by the formula

$$R = \sum_{i,j,k} [a^k_{ii} a^k_{jj} - (a^k_{ij})^2].$$

Since $R > 0$, it follows that

$$\sum_k [\sigma(\xi_k)]^2 = \sum_k \left(\sum_i a^k_{ii} \right)^2 = \sum_{i,j,k} a^k_{ii} a^k_{jj} \geqslant R > 0,$$

and hence $\sigma(\xi_k) \neq 0$ for some k.

Blanuša [25], [19], [20] showed that a real projective space $\mathbb{R}P^l$ with the standard metric can be isometrically embedded in E^N ($N = l(l+3)/2$). If we consider orthogonal coordinates in the Euclidean space, then the sphere S^l is given by the equation

$$\sum_{i=1}^{l+1} (x_i)^2 = r^2.$$

In this case an isometric embedding of $\mathbb{R}P^l$ in E^N ($N = n(n+3)/2$) is given in the form

$$x_{ik} = \frac{1}{r\sqrt{2}} x_i x_k \qquad (i = 1, \dots, l+1), \tag{3.3.4}$$

where the x_{ik} are orthogonal coordinates in E^N. This is a multidimensional generalization of the Veronese surface, and the submanifold thus obtained is the standard minimal isometric embedding of the sphere, corresponding to the second eigenvalue λ_2. We recall that minimal isometric embeddings of projective spaces in spheres correspond to even eigenvalues, and all the motions of $\mathbb{R}P^l$ are then induced by motions of the enveloping space. Under this embedding the geodesic curves are circles in the Euclidean space. If one adds this extra condition to the isometry condition, then the above estimate for the dimension of the enveloping space is the best possible, and an embedding is unique up to a motion of the enveloping space. This embedding is free, that is, the vectors $\dfrac{\partial r}{\partial u_i}$ and $\dfrac{\partial^2 r}{\partial u_i\,\partial u_j}$ are linearly independent at any point, where $r = r(u)$, $u = (u_1, \dots, u_l)$, is a local parametrization of the submanifold. This means that the dimension of the second osculating space is equal to $(l+3)l/2$. Other examples of isometric embeddings of $\mathbb{R}P^l$ in a Euclidean space are constructed in [25]. These are based on a generalization of the embedding (3.3.4), and in these cases all the motions of $\mathbb{R}P^l$ are again induced by motions of the enveloping space.

The lens space $L^n(r^m)$ is the quotient space of the sphere S^{2n+1} by a free action of the cyclic group Z/r^m given in the form

$$\xi^k z = \{\xi^k z_0, \dots, \xi^k z_n\},$$

where $\xi = \exp\left(i\pi/r^{m-1}\right)$ is a generator of the group Z/r^m.

Theorem 3.3.11 (Bernard [16]). *For $m \geqslant [\log_2 n] + [n/2]$ the lens space $L^n(r^m)$ admits no isometric immersion into $E^{4n-\alpha(n)}$, where $\alpha(n)$ is the number of* 1*s in the binary expansion of n.*

CHAPTER 4

ISOMETRIC IMMERSIONS OF PSEUDO-RIEMANNIAN SPACES OF CONSTANT CURVATURE

Let E^l_s be a real vector space with an inner product of signature $(s, l-s+1)$ given by the formula

$$\langle X, Y\rangle = -\sum_{i=1}^{s} x_i y_i + \sum_{j=s+1}^{l+1} x_j y_j \tag{4.0.1}$$

for vectors $X = (x_1, \dots, x_{l+1})$ and $Y = (y_1, \dots, y_{l+1})$, and let $c = \text{const} > 0$.

The hypersurface $S^l_s(c)$ determined by the formula

$$\langle X, X\rangle = \frac{1}{c^2} \tag{4.0.2}$$

is a pseudo-Riemannian space of constant positive curvature c and of signature $(s, l-s)$.

The hypersurface $H^l_s(c)$ given by the formula

$$\langle X, X\rangle = -\frac{1}{c^2}$$

is a pseudo-Riemannian space of signature $(s-1, l-s+1)$ and of constant negative curvature $-c$.

If $s = 0$, then $S^l_0(c) = S^l(c)$ is the standard sphere in the Euclidean space, and if $s = 0$, then $H^l_0(c)$ is the Lobachevsky space H^l. By changing the sign of the inner product (4.0.1) in E^{l+1}_s, we obtain the inner product in E^{l+1}_{l+1-s}, and $S^l_s(c)$ passes into $H^l_{l-s}(-c)$. Therefore, all results about the spheres $S^l_s(c)$ pass into assertions about the hyperbolic spaces $H^l_{l-s}(-c)$. Any totally geodesic m-dimensional submanifold of S^l_s (H^l_s) is a connected component of the intersection of S^l_s (H^l_s) with an $(m+1)$-dimensional linear subspace E^{l+1}_s (E^{n+1}_{s+1}). The sphere S^l_s is diffeomorphic to $S^{l-s} \times E^s$, and H^l_s is diffeomorphic to $S^s \times E^{l-s}$. The sphere S^2_1 with a metric of signature $(1,1)$ in the pseudo-Euclidean space E^3_1 is of positive curvature, and it is a one-sheeted hyperboloid of revolution in E^3_1, while $H^2_0 = H^2$ is a two-sheeted hyperboloid in which each sheet is isometric to the Lobachevsky space. However, there is no surface of class C^1 in E^3_1 with induced complete Riemannian metric of constant positive Gaussian curvature. Moreover, a closed two-dimensional manifold with metric of constant signature admits no C^1 isometric immersion into E^3_1 [158].

There is no regular extrinsically complete surface in E^3_1 with indefinite metric and constant negative curvature such that the limit cone of this surface is smooth everywhere except for the vertex [157].

For an isometric immersion of the Lobachevsky plane in E^3_1, the uniqueness problem (up to a motion in E^3_1) arises naturally. It turned out that for uniqueness up to parallel displacement and reflection one must fix not only a metric and a limit generator but also a ruled surface that approximates the given surface at infinity with higher accuracy than the limit cone. Then there is a unique extrinsically complete convex surface with metric of constant negative curvature and with isotropic limit cone with given approximation by a ruled surface at infinity [129].

Remark. Extrinsic completeness does not imply completeness of the metric of a surface. Therefore, the uniqueness problem for an isometric immersion of the complete Lobachevsky plane without asymptotics at infinity remains open in my opinion.

4.1. Immersions of zero extrinsic curvature. The local and global structure of isometric immersions of a pseudo-Euclidean (Lorentz) space E^l_1 into E^{l+1}_1 was studied by Graves [76]. In this case the estimate for the external null-index is the same as for a Euclidean space, namely, $\mu \geqslant l-1$. An analogue of Lemma 3.1.1 remains valid; however, completeness of a leaf is understood as geodesic completeness.

A vector X in E^{l+1}_1 for which $\langle X, X\rangle = 1$ is said to be *space-like*, and if $\langle X, X\rangle = -1$, then the vector is said to be *time-like*. If $\langle X, X\rangle = 0$, then the vector is said to be *light-like*, or a *null-vector*. A *null-frame* in E^3_1 is a triple of vectors

$$F = (A, B, C) = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix},$$

where A and B stand for null-vectors for which $\langle A, B\rangle = -1$ and C for a unit space-like vector orthogonal to the plane spanned by the vectors A and B, and $\det F = +1$. A regular curve $s \to F(s)$ in the connected component of the group of motions of E^3_1 is said to be a *reference curve.* For a reference curve, the following Frenet equations hold:

$$\frac{dA}{ds} = k_1(s)\,A(s) + k_2(s)C(s),$$
$$\frac{dB}{ds} = -k_1(s)\,B(s) + k_3(s)C(s),$$
$$\frac{dC}{ds} = k_3(s)\,A(s) + k_2(s)B(s).$$

A curve $x(s)$ in E^3_1 is called a *null-curve* if one has $\left\langle \frac{dx}{ds}, \frac{dx}{ds} \right\rangle = 0$ at any point, where $\langle\,\cdot\,,\cdot\,\rangle$ is the inner product in E^3_1.

If $x(0) = 0$, then the null-curve belongs to the isotropic cone. For a null-curve one can choose a null-frame in such a way that $\frac{dx}{ds}$ is proportional to $A(s)$. One can choose a vector $C(s)$ (similar to the principal normal of a curve in the Euclidean case) in such a way that the Frenet equations become

$$\frac{dx}{ds} = A(s), \quad \frac{dA}{ds} = k_2(s)C(s), \quad \frac{dB}{ds} = k_3(s)C(s), \quad \frac{dC}{ds} = k_3(s)A(s) + k_2(s)B(s).$$

Such a frame is called a *Cartan frame* for the null-curve. Let us consider a ruled (B-ruled) surface with the parametrization

$$f(s, u) = x(s) + uB(s).$$

This surface is endowed with the induced indefinite (Lorentz) metric; it is isometric to the Lorentz plane E^2_1 only if $k_3(s) = 0$, that is, $\frac{dB}{ds} = 0$ and $B(s)$ is a parallel isotropic vector field. In this case we refer to a B-ruled surface as an *isotropic cylinder*. We note that, in contrast to the Euclidean case, the directrix of this cylinder belongs to the isotropic cone rather than to a two-dimensional plane.

If an isometric regular immersion $f\colon E^l_1 \to E^{l+1}_1$ is not totally geodesic, then there is a point P with external null-index $\mu = l-1$, and the null-index is constant in a neighbourhood of this point. It follows from an analogue of Lemma 3.1.1 that the image $F^l_1 = f(E^l_1)$ contains a complete ruled subspace $T_{l-1}(P)$ of the enveloping space. The following cases are possible:

1) $T^{l-1}(P)$ is a Lorentz subspace of E^{l-1}_1;
2) $T^{l-1}(P)$ is a Euclidean subspace of E^{l-1};
3) $T^{l-1} = E^{l-2} \oplus \mathrm{Span}\{\xi\}$, where E^{l-2} is the Euclidean space, ξ is a null-vector, and $\langle y, \xi\rangle = 0$ for any $y \in E^{l-2}$.

In the cases 1) and 2) the metric of $T^{l-1}\,(P)$ is non-degenerate and in the case 3) it is degenerate.

The following analogue of the Hartman–Nirenberg theorem holds [82].

Theorem 4.1.1 (Graves [76]). *A regular isometric immersion $f\colon E^l_1 \to E^{l+1}_1$ has one of the following forms.*

1) $E^l_1 = E^{l-1}_1 \times E^1 \overset{\mathrm{id}\times c}{\longrightarrow} E^{l-1}_1 \times E_2$, where c is a plane curve in the Euclidean plane. This surface is a cylinder over a plane Euclidean curve with generator E^{l-1}_1.

2) $E^l_1 = E^{l-1} \times E^1_1 \overset{\mathrm{id}\times c}{\longrightarrow} E^{l-1} \times E^2_1$, where c is a plane time-like curve in a pseudo-Euclidean plane. This surface is a cylinder over a plane time-like curve with a Euclidean generator E^{l-1}.

3) $E^l_1 = E^{l-2} \times E^2_1 \overset{\mathrm{id}\times g}{\longrightarrow} E^{l-2} \times E^3_1$, where $g\colon E^2_1 \to E^3_1$ is an isotropic cylinder. This is a cylinder over an isotropic cylinder in E^3_1 with a Euclidean generator E^{l-2}.

All products are orthogonal in the metric of the enveloping space. The hypersurface $f(E^l_1) \subset E^{l+1}_1$, which is isometric to E^l_1, is a cylinder with $(l-1)$-dimensional generator. The generators of this cylinder can be

1) a Lorentz subspace E^{l-1}_1,
2) a Euclidean subspace E^{l+1},
3) a subspace $E^{l-2} \oplus \operatorname{Span}\{\xi\}$ with degenerate metric.

The case of an isometric embedding of E^l_s in E^{l+p}_{s+k} with external null-index $\mu \geqslant l-1$ was considered in [4], and results similar to Theorem 4.1.1 were obtained. However, without an external restriction on the null-index, the problem of the structure of a submanifold $F^l \subset E^{l+p}_s$ isometric to E^l_s remains open.

The next assertion follows from the papers [36] and [38].

Theorem 4.1.2. *Let $f\colon E^l_s \to E^{l+p}_s$ be an isometric immersion with constant external null-index. If at a point Q_0 the null-space $L^k(Q_0)$ is a pseudo-Euclidean plane E^k_s, then the submanifold $f(E^l_s)$ is a cylinder with generator E^{l-p}_s.*

The following analogue of Lemma 2.1.1 holds for the pseudo-Euclidean space E^n_1.

Lemma 4.1.1 ([36], [38]). *Suppose that M^2_1 is a surface of class C^1 in a pseudo-Euclidean space E^n_1 and M^2_1 is isometric to a strip on the Lorentz plane E^2_1 between parallel time-like lines. If there is a time-like line of the enveloping space that belongs to the surface M^2_1, then the surface is a cylinder with time-like generator.*

Proof. Let $\overline{l}$ be a line on the surface M^2_1, and let l be the corresponding line on E^2_1, $\overline{l} = \phi(l)$, where ϕ is an isometry between E^2_1 and M^2_1. We parametrize l in some way by $\gamma(t)$, $t \in \mathbb{R}$, and for any t we denote by λ_t a segment orthogonal to l at the point $\gamma(t)$. We claim that $\phi(\lambda_t) = \overline{\lambda}_t$ belongs to the subspace of E^n_1 orthogonal to $\overline{l}$ with respect to the pseudo-Euclidean metric of E^n_1. We assume that this fails at some point $\gamma(t_0) = q_0$; without loss of generality one can assume that $t_0 = 0$. By assumption, there is a point $p \in E^2_1$ such that the segment $\overline{q}_0\overline{p}$ in E^n_1 is not orthogonal to $\overline{l}$ in the enveloping space, where $\overline{q}_0 = \phi(q_0)$ and $\overline{p} = \phi(p)$. We denote the length of the segment q_0p_0 by h and write $\overline{\gamma}(t) = \phi(\gamma(t))$. Suppose that t is so large that the sides of the triangle $\gamma(-t)p\gamma(t)$ are time-like. If t is the length of the line l from the point $q_0 = \gamma(0)$ to the point $\gamma(t)$, then

$$|p\gamma(t)|^2 = |p\gamma(-t)|^2 = |-t^2 + h^2| = t^2 - h^2.$$

Let us choose an orthogonal system of coordinates in E_1^n in such a way that the line $\overline{l}$ coincides with the axis x_0, the point $\overline{q}_0$ is the origin, and the point $\overline{p}$ belongs to the plane x_0, x_1. Then the coordinates of the point $\overline{p}$ are $(t_0, \overline{h}, 0, \dots, 0)$ with $t_0 > 0$ (or < 0), and

$$|\overline{p}\,\overline{\gamma}(t)|^2 = |-(t-t_0)^2 + \overline{h}^2| = (t-t_0)^2 - \overline{h}^2,$$
$$|\overline{p}\,\overline{\gamma}(-t)|^2 = |-(t+t_0)^2 + \overline{h}^2| = (t+t_0)^2 - \overline{h}^2.$$

Since ϕ is an isometry, it follows that $t^2 - h^2 = d(\overline{p}, \overline{\gamma}(t))^2$, where d is the distance on M_1^2. Since a time-like line has the greatest length among all time-like curves joining two chosen points, we see that

$$(t-t_0)^2 - \overline{h}^2 \geqslant t^2 - h^2,$$
$$(t+t_0)^2 - \overline{h}^2 \geqslant t^2 - h^2.$$

Hence,

$$-2tt_0 + t_0^2 \geqslant \overline{h}^2 - h^2,$$
$$2tt_0 + t_0^2 \geqslant \overline{h}^2 - h^2,$$

which is impossible for large t. We arrive at a contradiction.

Let us consider a quadrangle $q_0\,\gamma(t)\,p(t)\,p_0$ on the plane E_1^2, where $q_0 \in l$ and the segments g_0p_0 and $\gamma(t)\,p(t)$ are orthogonal to l. The corresponding quadrangle in E_1^n is $\overline{q}_0\,\overline{\gamma}(t)\,\overline{p}(t)\,\overline{p}_0$. We take in E_1^n the above system of coordinates, in which $\overline{p}_0 = (0, \overline{h}_0, 0, \dots, 0)$ and $\overline{p}(t) = (t, \overline{h}, \cos\alpha, \overline{h}, \sin\alpha, 0, \dots, 0)$, where α is the angle between $\overline{g}_0\,\overline{p}_0$ and $\overline{\gamma}(t)\,\overline{p}(t)$.

As we know,

$$|\overline{p}_0\,\overline{p}(t)|^2 = t^2 - (\overline{h}_0 - \overline{h}_1\,\cos\alpha)^2 - (\overline{h}_1\,\sin\alpha)^2.$$

Since ϕ is an isometry, it follows that

$$|\overline{p}_0\,\overline{p}(t)|^2 \geqslant |p_0\,p(t)|^2 = t^2.$$

Hence, $t^2 - (\overline{h}_0 - \overline{h}_1\,\cos\alpha)^2 - (\overline{h}_1\,\sin\alpha)^2 \geqslant t^2$, $\overline{h}_0 = \overline{h}_1$, $\alpha = 0$, and $|\overline{p}_0\,\overline{p}(t)| = t$. This means that the interval $\overline{p}_0\,\overline{p}(t)$ belongs to the surface M_1^2 and is parallel to the line $\overline{l}$.

4.2. Isometric immersions of pseudo-Riemannian spaces of non-zero constant curvature. Isometric immersions of pseudo-Riemannian spaces $S_s^l(1)$ of constant positive curvature 1 into $S_{s+t}^{l+p}(1)$ were considered in [52]–[54] and [77]. Conditions for an isometric immersion to be totally geodesic were found. There is a result for standard spherical spaces which is similar to Theorem 3.1.2.

Theorem 4.2.1 (Dajczer and Fornari [53]). *Let $f\colon S^l_s(1) \to S^{l+p}_s(1)$ be a regular isometric immersion. If $p < l - s$, then $f(S^l_s(1))$ is a totally geodesic submanifold.*

An example of an isometric immersion $S^2_1 \to S^3_1$ (for $s = 1$, $l = 2$, and $p = 1$) that is not totally geodesic was constructed in [77].

Under an additional condition on the null-space one has the following corollary to a more general theorem.

Theorem 4.2.2 (Abe and Magid [3]). *Let $f\colon S^l_s(c) \to S^{l+p}_s(c)$ be an isometric immersion of pseudo-Riemannian space forms. Let the null-space L^k be a pseudo-Euclidean plane E^k_s for a point with minimal null-index. If $p < l$, then $F^l_s = f(S^l_s(c))$ is a totally geodesic submanifold of $S^{l+p}_s(c)$.*

A similar theorem for an isometric immersion $f\colon S^l(c) \to S^{l+p}_p(c)$ follows from an earlier paper [33]. If $p < l$, then $F^l = f(S^l)$ is a totally geodesic standard submanifold.

Theorem 4.2.3 (Dajczer and Dombrowski [52]). *Let $f\colon S^l_s(1) \to S^{l+p}_{s+1}(1)$ be a regular isometric immersion with non-degenerate null-space of minimal dimension. If $p < \frac{1}{2}(l - s)$, then $f(S^l_s(1))$ is a totally geodesic submanifold of S^{l+p}_{s+1}.*

The theorem fails without the additional conditions. There are isometric immersions $f\colon S^l_s(1) \to S^{l+2}_{s+1}$ for $0 \leqslant s \leqslant l-1$ that are not totally geodesic submanifolds [53]. A description of such immersions is given in [54].

The point is that, if the signatures related to the metrics of the embedded submanifold and the enveloping space are distinct, then the normal bundle is Lorentzian rather than Riemannian, and the image of the second fundamental form can be contained in a null-direction, in which case we have no lower bound for the external null-index.

A complete description of isometric immersions of the pseudo-Riemannian space H^l_1 of constant negative curvature -1 into the space H^{l+1}_1 of the same curvature in given in [1] using the language of geodesic laminations. This description is similar to that for the isometric immersions of the Lobachevsky space $H^l(-1)$ into $H^{l+1}(-1)$ (Theorem 3.1.6).

Local and global isometric immersions of a pseudo-Riemannian space of constant negative curvature H^l_s in a pseudo-Euclidean space E^n_s were studied in [32].

Theorem 4.2.4 [32]. *Let H^l_s be a pseudo-Riemannian space of constant negative curvature, and let E^n_s be a pseudo-Euclidean space. If $n < 2l-1$, then the manifold H^l_s cannot be immersed locally isometrically into E^n_s; if $n = 2l-1$, then a locally isometric immersion exists.*

For $s = 0$ one obtains the following known theorem: An l-dimensional Lobachevsky space cannot be embedded locally isometrically in the Euclidean space E^{2l-2}, and there is a locally isometric embedding in E^{2l-1}. This theorem was discovered by Cartan in 1919; in 1938 it was rediscovered by Liber; an example of a local embedding was constructed by Shur in 1886. The next assertion partially answers the question of global immersion of a complete pseudo-Riemannian space of constant negative curvature.

Theorem 4.2.5 [32]. *Let H_s^l be a complete connected pseudo-Riemannian manifold of constant negative curvature. If $s \neq 0, 1, 3, 7$, then the manifold H_s^l cannot be isometrically immersed into E_s^{2l-1}.*

There are reasons to believe that Theorem 4.2.5 is valid for $s = 0, 1, 3, 7$ as well, but a proof in this case is not known to the author. Theorems 4.2.4 and 4.2.5 can be naturally generalized to the case when the enveloping space is a pseudo-Riemannian manifold of constant curvature.

Theorem 4.2.6 [32]. *Let H_s^l be a complete connected pseudo-Riemannian manifold of constant curvature c, let H_s^{2l-1} be a complete connected pseudo-Riemannian manifold of constant curvature C, and let $c < C$. If the manifold H_s^l is a regular surface in H_s^{2l-1} and if $s \neq 0, 1, 3, 7$, then $C > 0$, $c = 0$, and the manifold H_s^l is locally isometric to a pseudo-Euclidean space.*

We note that a theorem related to Theorem 3.2.9 holds in a Riemannian space. For a pseudo-Riemannian space one has the following result.

Theorem 4.2.7 (Wolf [175]). *Let H_s^l be a complete connected pseudo-Riemannian manifold of constant curvature K. A universal covering manifold of H_s^l is given by the hypersurface*

a) $\sum_{i=1}^{p+1} x_i^2 - \sum_{j=1}^{s} x_{p+1+j}^2 = K^{-1/2}$ *in the space* E_s^{l+1} *for* $p \neq 1$ *and* $K > 0$,

b) $\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{s+1} x_{p+j}^2 = -K^{-1/2}$ *in the space* E_s^{l+1} *for* $s \neq 1$ *and* $K < 0$,

and the pseudo-Euclidean space E_s^l *for* $K = 0$.

Proof of Theorem 4.2.4. Suppose the contrary. Let H_s^l be a regular l-dimensional surface of constant negative sectional curvature in E_s^{2l-2}. Without loss of generality one can assume that the curvature is equal to -1 for any two-dimensional plane. Let Q be an arbitrary point of the surface H_s^l, let T_Q be the tangent space at Q, let $x = (x^1, \ldots, x^l)$ be an arbitrary vector in T_Q, and let $g = g_{ij}x^i x^j$ be the metric form of the surface at Q. The rank of the fundamental form g is equal to l at any point. Let $A^m = A_{ij}^m x^i x^j$ be the second fundamental form of H_s^l at Q with respect to a normal n_m ($m = 1, \ldots, l-2$). Since all normals of the surface are space-like vectors, we can take the normals n_m to be mutually orthogonal and with unit norm. By the Gauss formula, the curvature tensor of the surface H_s^l is equal to

$$R_{ijkl} = \sum_m (A_{ik}^m A_{ji}^m - A_{il}^m A_{jk}^m). \tag{4.2.1}$$

Since H_s^l is a space of constant negative curvature, it follows that

$$R_{ijkl} = -(g_{ik}g_{ji} - g_{il}g_{jk}). \tag{4.2.2}$$

It follows immediately from the formulae (4.2.1) and (4.2.2) that

$$\sum_m (A_{ik}^m A_{ji}^m - A_{il}^m A_{jk}^m) + (g_{ik}g_{ji} - g_{il}g_{jk}) = 0. \tag{4.2.3}$$

Let us consider the system of $l-1$ equations

$$\begin{aligned} &A^m_{ij}x^i x^j = 0 \quad (m = 1, \dots, l-2),\\ &g_{ij}x^i x^j = 0. \end{aligned} \tag{4.2.4}$$

It follows from the conditions (4.2.3) that the system (4.2.4) satisfies the assumptions in Lemma 3.1.1. Hence, the system (3.2.1) has a non-trivial solution x_0. Then it follows from the equations (4.2.3) that

$$-\sum_m (A^m(x_0, y))^2 - g^2(x_0, y) = 0 \tag{4.2.5}$$

for any $y \in T_Q$.

It follows from (4.2.5) that $g(x_0, y) = 0$ for any $y \in T_Q$. This means that the rank of the fundamental form g is less than l. We arrive at a contradiction.

If $n = 2l-1$, then a locally isometric embedding of the manifold H^l_s of constant negative curvature in the space E^{2l-1}_s can be defined as follows:

$$y_{2i-1} = \begin{cases} a_i e^u_e \cos \dfrac{u_i}{a_i}, & 1 \leqslant i \leqslant p-1;\\ a_i e^u_e \cosh \dfrac{u_i}{a_i}, & p \leqslant i \leqslant l-1; \end{cases}$$

$$y_{2i} = \begin{cases} a_i e^u_e \sin \dfrac{u_i}{a_i}, & 1 \leqslant i \leqslant p-1;\\ a_i e^u_e \sinh \dfrac{u_i}{a_i}, & p \leqslant i \leqslant l-1; \end{cases}$$

$y_{2l-1} = \displaystyle\int_0^{u_e} \sqrt{1-e^{2u}}$, where $y_1, \dots, y_{2l-1}$ are the coordinates in E^{2l-1}_s and a_1, $\dots$, a_l are real numbers satisfying the condition $\sum_i a_i^2 = 1$.

In the proof of Theorem 4.2.5 it suffices to restrict ourselves to the case in which H^l_s is a complete simply connected pseudo-Riemannian manifold of constant negative curvature. Indeed, if a complete manifold H^l_s is isometrically immersed into the space E^{2l-1}_s, then the universal covering manifold is also isometrically immersed into E^{2l-1}_s.

Proof of Theorem 4.2.5. Suppose the contrary. Let H^l_s be a complete simply connected surface of constant negative curvature in E^{2l-1}_s. It follows from the equations (4.2.4) that the symmetric bilinear forms $A^m = A^m_{ij}x^i y^j$ and $g = g_{ij}x^i y^j$ $(m = 1, \dots, l-1)$ are exteriorly orthogonal at any point of the surface. Since the rank of the form g is equal to l, it follows that the bilinear forms A^m and g satisfy the conditions of the Cartan theorem (Theorem 3.2.3). Therefore, at any point Q of H^l_s there is a unique basis in the tangent space T_Q in which the forms A^m and g are diagonal. This basis is formed by orthogonal vectors, say, $e_1, \dots, e_{p+s}$, because the form g is diagonal. We note that the vectors $e_1, \dots, e_p$ are space-like and the vectors $e_{p+1}, \dots, e_{p+s}$ are time-like, because their norms satisfy the conditions $g_{ij}e^i_k e^j_k > 0$

for $k \leqslant p$ and < 0 for $k > p$. By the Wolf theorem, the manifold H_s^l is globally isometric to the hypersurface H_s^l given by the equation

$$\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{s+1} x_{p+j}^2 = -1$$

in the pseudo-Euclidean space E_{s+1}^{l+1}. Let us consider the sphere S^s that belongs to the hypersurface H_s^l and is given by the equations $x_i = 0$ ($i = 1, \dots, p$). Let $T^s \subset T_Q$ be the tangent space of S^s at Q. This space consists of time-like vectors. We choose the following orthogonal basis in the tangent space T_Q: the time-like vectors $r_{p+1}, \dots, r_{p+s}$ form an orthogonal basis in the plane T^s and the space-like vectors $r_1, \dots, r_p$ are taken from the orthogonal complement of T^s. Then

$$l_{p+j} = \sum_{i=1}^{l} \lambda_{p+j}^i r_i \quad (j = 1, \dots, s).$$

We claim that

$$\det(\lambda_{p+j}^{p+i}) \neq 0 \quad (i, j = 1, \dots, s). \tag{4.2.6}$$

Suppose not. Then there is a non-zero vector $e = \sum_{j=1}^{s} \mu_j e_{p+j} = \sum_{i=1}^{p} \nu_i r_i$ that is simultaneously a linear combination of the independent time-like vectors e_{p+j} and a linear combination of independent space-like vectors r_i ($i = 1, \dots, p$).

Therefore, the non-zero vector e is simultaneously time-like and space-like, which is impossible. We take the orthogonal projection of the vector e_{p+j} ($j = 1, \dots, s$) on the subspace T^s in the sense of the pseudo-Euclidean metric of the tangent space T_Q. Since the determinant (4.2.6) is non-zero, it follows that the vectors e_{p+j} are mapped to linearly independent vectors e'_{p+j} tangent to the sphere S^s. These vectors parallelize the sphere S^s. By the Adams theorem, only the spheres S^1, S^3, and S^7 are parallelizable. We arrive at a contradiction to the assumption.

Proof of Theorem 4.2.6. The complete simply connected surface H_s^l is a surface of constant negative extrinsic curvature in the manifold H_s^{2l-1}. By the Gauss formula, the curvature tensor of H_s^l is equal to

$$R_{ijkl} = \sum_{m=1}^{l-1} (A_{ik}^m A_{jl}^m - A_{il}^m A_{jk}^m) + C(g_{ik} g_{jl} \cdot g_{il} g_{jk}). \tag{4.2.7}$$

It follows from formulae (4.2.2) and (4.2.7) that the bilinear forms A^m ($m = 1, \dots, l-1$) and g are exteriorly orthogonal. The unique basis in which the forms are diagonal parallellizes the surface H_s^l. The subsequent part of the proof of the fact that the case $c < 0$ is impossible is similar to that in Theorem 4.2.5. When proving that the case $c > 0$ is impossible, one must take into account that the surface H_s^l is globally isometric to the hypersurface

$$\sum_{i=1}^{p+1} x_i^2 - \sum_{j=1}^{s} x_{p+1+j}^2 = c^{-1/2}$$

in the pseudo-Euclidean space $E^{l+1}_{s'}$. In the proof we parallelize the sphere S^p given by the equations $x_{p+1+j} = 0$ $(j = 1, \dots, s)$. Therefore, we obtain conditions on p as well.

Let S^{2l-1}_s be the hypersurface in E^{2l}_s given by the equation

$$\sum_{k=1}^{2p} x_k^2 + \sum_{j=1}^{s} x_{2p+2j-1}^2 - \sum_{j=1}^{s} x_{2p+2j}^2 = 1.$$

Let H^l_s be an l-dimensional complete surface in S^{2l-1}_s given by the equations

$$x_{2i-1} = \begin{cases} \dfrac{1}{\sqrt{2l}} \cos u_i & \text{for } 1 \leqslant i \leqslant p; \\ \dfrac{1}{\sqrt{2l}} \cosh u_i & \text{for } p+1 \leqslant i \leqslant l; \end{cases}$$

$$x_{2i} = \begin{cases} \dfrac{1}{\sqrt{2l}} \sin u_i & \text{for } 1 \leqslant i \leqslant p; \\ \dfrac{1}{\sqrt{2l}} \sinh u_i & \text{for } p+1 \leqslant i \leqslant l. \end{cases}$$

The sectional curvature of the surface V^l_s is identically zero, and the curvature of the hypersurface S^{2l-1}_s is equal to 1. This example realizes the case admitted by Theorem 4.2.6.

Similar theorems hold for the dual space of constant positive curvature.

CHAPTER 5

SOLITONS AND GEOMETRY

5.1. Solitons and pseudo-spherical surfaces. The equations of isometric immersions of space forms of constant curvature into spaces of constant curvature have the same properties as the soliton equations; in particular, they have a Lax pair and Bäcklund transformations [163].

In general, the first soliton equation, namely, the sine-Gordon equation $\omega_{xy} = \sin\omega$, arose in geometry in the paper of Chebyshev "On dress-making" [46] in 1878.

Poznyak proved [136], [137] that any solution $\omega(x,y)$ of the sine-Gordon equation generates a surface in E^3 of Gaussian curvature $K = -1$ on which $\omega(x,y)$ is the angle between the asymptotic curves, x and y are the arclength parameters on these curves, and the surface has singularities at the points where $\sin\omega = 0$. Other soliton equations, in particular, the Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0$$

and the modified Korteweg–de Vries equation

$$u_t + 6u^2u_x + u_{xxx} = 0$$

also arose on surfaces of constant negative curvature. Namely, under a special choice of the coordinate system (x, t) on a surface of constant negative curvature, the geodesic curvatures $u(x, t)$ of the family of coordinate curves $t = \text{const}$ satisfy the above equations; see [49], [50], [148]. The sine-Gordon equation and other soliton equations arise in the same way.

For an isometric immersion of two-dimensional metrics of constant Gaussian curvature c into three-dimensional space of constant curvature $\widetilde{c}$, $c \neq \widetilde{c}$, the integrability conditions reduce to the sine- and sinh-Gordon equations. For an isometric immersion of $M^l(c)$ into $M^{l+p}(\widetilde{c})$, the Gauss, Codazzi, and Ricci integrability equations are regarded as a multidimensional generalization of the sine-Gordon equation. For $c < 0$ and $\widetilde{c} = 0$, the Codazzi–Ricci equations have the Gauss equation as an integrability condition. Therefore, for $c = -1$, $\widetilde{c} = 0$, and $p = l - 1$ the entire system of immersions reduces to the system (3.2.9). By using the soliton system, isometric immersions of finite type with flat normal connection of the form $f \colon M^l(c) \to M^{l+p}(\widetilde{c})$ were obtained, where $c \neq 0 \neq \widetilde{c}$ and $c \neq \widetilde{c}$ [71].

The main contribution of the classical geometers to soliton theory is in the construction of explicit solutions of soliton equations [164].

In 1883, Bäcklund introduced a transformation that for a surface of constant negative curvature in E^3 constructs another surface of constant negative curvature. A Bäcklund map between two surfaces F_1 and F_2 has the property that the line interval joining two corresponding points is tangent to both surfaces. If 1) the intervals have constant length and 2) the angle between the normals at the corresponding points is constant, then the surfaces F_1 and F_2 have constant negative Gaussian curvature.

The Bäcklund transformation in the multidimensional case of an immersion of the Lobachevsky space L^l into E^{2l-1} (or a *pseudo-spherical congruence transformation*) was introduced and studied by Tenenblat and Terng [161]. For a given submanifold $F^l \subset E^{2l-1}$ of constant negative curvature this transformation enables one to construct some other submanifold $\widetilde{F}^l \subset E^{2l-1}$ of constant negative curvature. Thus, for a given soliton solution of the system (3.2.9) one can construct a new solution, and the asymptotic curves are taken to asymptotic curves.

Suppose that multidimensional submanifolds F_1^l and F_2^l of E^{2l-1} satisfy the tangency condition. If the condition 1) is preserved and the condition 2) is replaced by the assumption that the isoclinic angles between the normal spaces at the corresponding points are constant, then the submanifolds F_1^l and F_2^l of E^{2l-1} are of constant negative sectional curvature. Under this transformation, the principal curves (and the asymptotic curves) also correspond to curves of the same type [162]. A system analogous to (3.2.9) for submanifolds of constant sectional curvature in pseudo-Riemannian space forms was studied in [15], and the Bäcklund transformation for this system was investigated in [43].

A multidimensional generalization of the Bianchi transformation was given in [8] by Aminov. Let F^l be a surface of constant negative sectional curvature in E^{2l-1}. Let $u = (u_1, \dots, u_l)$ be semigeodesic coordinates on F^l such that the metric of F^l can be represented in the form

$$ds^2 = du_l^2 + e^{2u_l}(du_1^2 + \dots + du_{l-1}^2).$$

Then the Bianchi transformation sends a point with radius vector $r = r(u)$ to the point

$$\overline{r} = r - \frac{\partial r}{\partial u_l}\,.$$

The Bianchi transformation, as well as the Bäcklund transformation, maps a submanifold F^l of constant negative curvature in E^{2l-1} to a submanifold of the same constant negative curvature [8], and thus constructs a new solution of the system (3.2.9) from a known solution of this soliton system. Bianchi transformations were also considered in [115].

5.2. Isometric immersions with additional constraints. If $f\colon M^l(c) \to M^{l+p}$ is an isometric immersion into $M^{l+p}(\widetilde{c})$ with flat normal connection and if f is minimal, then the submanifold $f(M^l(c))$ is a part of a Clifford torus embedded in the sphere S^{2l-1} in the standard way [51], [117]. If the minimality condition is replaced by the parallel condition for the vector of mean curvature, then the following assertions hold:

1) for $c = \widetilde{c}$ either f is totally geodesic or $c = 0$ and

$$f(M^l) \subset S^1(r_1) \times \cdots \times S^1(r_p) \times E^{l-p} \subset E^{l+p};$$

2) for $\widetilde{c} > c = 0$

$$f(M^l) \subset S^1(r_1) \times \cdots \times S^1(r_l) \times S^{2l-1}(\widetilde{c}) \subset E^{2l},$$

where $r_1^2 + \cdots + r_l^2 = 1/\widetilde{c}$;

3) for $\widetilde{c} < c = 0$

$$f(M^l) \subset H^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_l) \subset H^{2l-1}(\widetilde{c}) \subset E_1^{2l},$$

where $-r_1^2 + r_2^2 + \cdots + r_l^2 = 1/\widetilde{c}$;

4) $f = i \circ f'$, where f' is of type 1), 2), or 3) and i is an umbilic or a totally geodesic inclusion [56].

Lagrangian isometric embeddings of a real space form $M^l(c)$ in a complex space form $\widetilde{M}^l(4c)$ are determined in [47].

The Lagrangian minimal submanifolds of constant sectional curvature $c \neq \widetilde{c}$ in a complex indefinite space form $\widetilde{M}_1^n(4\widetilde{c})$ of constant holomorphic curvature $4\widetilde{c}$ are completely classified [106]. In a complex Riemannian space form, this result was obtained in [64]. In [121] all locally isometric immersions into complex space forms of complex Riemannian space forms of constant holomorphic curvature as Kähler submanifolds of these space forms are determined.

Theorem 5.2.1 (Nakagawa and Ogiue [121]). *Let $F^l(c)$ be a Kähler submanifold of constant holomorphic curvature $\widetilde{c}$ in a complex space form $M^{l+p}(\widetilde{c})$ of constant holomorphic curvature $\widetilde{c}$.*

1) *If $\widetilde{c} > 0$ and the embedding is complete (that is, the submanifold does not belong to a totally geodesic subspace of lesser dimension), then $\widetilde{c} = \nu c$ and $l + p = \binom{l+\nu}{\nu} - 1$, where ν is a positive integer.*

2) *If $\widetilde{c} \leqslant 0$, then $\widetilde{c} = c$, and $F^l(c)$ is a totally geodesic submanifold of $M^{l+p}(\widetilde{c})$.*

Some unsolved problems

1. Classical results concerning the unique determination of the sphere S^2 in E^3 and the fact that a complete surface of zero Gaussian curvature in E^3 is a cylindrical surface remain valid in the class of C^1 surfaces of bounded extrinsic curvature, and also in the class of C^1 affine stable immersions (Chapter 1).

However, it is not known whether the Hilbert theorem on the impossibility of an isometric immersion of the Lobachevsky plane into the Euclidean space E^3 holds in the class of C^1 surfaces of locally bounded extrinsic curvature [141].

The intrinsic geometry of Aleksandrov spaces has recently been actively investigated; these spaces are a non-regular generalization of Riemannian and Finsler spaces. However, non-regular analogues have not been developed in the extrinsic geometry of submanifolds, although one can introduce a definition of class C^1 submanifolds of bounded extrinsic curvature for arbitrary codimension and prove analogues of many theorems in the class of these submanifolds, and also in the class of affine stable immersions.

2. In [159] there is a thorough presentation on problems of isometric immersion of complete two-dimensional metrics of constant curvature into simply connected spaces M^3 of constant curvature (§2.2); it is mentioned there that it is not known whether there are complete regular surfaces in the Lobachevsky space $H^3(-1)$ with extrinsic curvature $0 < K_{\text{ext}} < 1$ (or, equivalently, with Gaussian curvature $-1 < K < 0$) that differ from equidistant surfaces and surfaces of revolution.

3. In [134], isometric immersions of complete two-dimensional metrics into the Euclidean space with minimal codimension are given explicitly (§2.3). All in all, there are ten flat three-dimensional space forms. It would be natural to extend the result to these forms.

4. In §2.3 the local structure of isometric immersions of flat two-dimensional metrics into the Euclidean space E^4 in the form of surfaces with flat normal connection is described. The problem is to clarify the global structure of compact regular two-dimensional surfaces of zero Gaussian curvature and with flat normal connection.

Conjecture. These are either flat tori in the spherical space S^3 or the Cartesian metric product of two plane closed curves that belong to complementary orthogonal planes.

A similar result must hold for compact submanifolds F^l of zero sectional curvature with flat normal connection in E^{2l}.

5. Find all two-dimensional surfaces of constant positive Gaussian curvature for which all intrinsic motions are induced by motions of the enveloping Euclidean space.

Conjecture. These are only standard isometric immersions with radius vector consisting of an orthonormal basis of eigenfunctions of the Laplace operator on the sphere that correspond to the same eigenvalue.

6. As is known (Tompkins theorem), there is no compact flat submanifold F^l of a Euclidean space E^{l+p}, $p < l$ (this follows from Theorem 3.1.1).

Let M^l and M^{l+p} be two flat space forms. In dependence on the ranks of the fundamental groups, estimate the codimension for which an isometric immersion is impossible.

Conjecture. Let r_1 be the rank of $\pi_1(M^l)$ and r_2 the rank of $\pi_1(M^{l+p})$. If $p < r_1 - r_2$, then there is no isometric immersion of M^l into M^{l+p}.

7. A complete surface F^l of zero sectional curvature in E^{l+p} is a cylinder with $(l-p)$-dimensional generator (Theorem 3.1.1).

The problem is to generalize this theorem to flat space forms. If M^l and M^{l+p} are compact and $f\colon M^l \to M^{l+p}$ is an isometric immersion, then M^l is locally a metric product, $M^l = M^k \times E^{l-k}$, $k \leqslant p$, where E^{l-k} is a Euclidean factor.

Conjecture. There is a covering $\widetilde{M}^l$ of finite multiplicity such that $\widetilde{M}^l$ is the Cartesian product $\widetilde{M}^l = \widetilde{M}^k \times T^{l-k}$, where $\widetilde{M}^k$ is a compact manifold and T^{l-k} is a torus.

Is it true that $\widetilde{M}^l$ is a metric product, where $\widetilde{M}^k$ and T^{l-k} are flat manifolds?

A similar result must also hold for non-compact flat space forms, where the dimension of the torus depends on the ranks of the fundamental groups of M^l and M^{l+p}.

8. A complete not simply connected hyperbolic space form admits no regular isometric immersion into E^{2n-1} (corollary to Theorem 3.2.8).

The multidimensional Hilbert problem is to prove this assertion in the simply connected case, that is, prove that there is no regular isometric immersion of the Lobachevsky space H^l into the Euclidean space E^{2l-1}.

There is a similar problem if the enveloping space is the Lobachevsky space.

9. Bieberbach constructed a geometric immersion of the Lobachevsky plane in a Hilbert space such that all motions of the Lobachevsky plane are induced by motions of the Hilbert space [17]. There is no such immersion of the Lobachevsky plane into any finite-dimensional Euclidean space [98]. Blanuša constructed an isometric embedding of the Lobachevsky space H^n in a Hilbert space [18]. Is it true that the motions of the Lobachevsky space H^n are induced by motions of the Hilbert space?

10. Moore proved that a compact submanifold F^l of constant positive sectional curvature in E^{l+p} is simply connected (that is, it is isometric to the standard sphere) if $p \leqslant l/2$ (Theorem 3.3.3).

Conjecture. This result holds for $p < l$, and possibly even for $p \leqslant l$.

11. Moore proved that an isometric immersion of the standard sphere S^l into E^{l+2} can be extended to an isometric immersion of the ball D^{l+1} bounded by S^l into E^{l+1} (Theorem 3.3.4). The problem is to generalize this theorem to the case in which the sphere S^l is replaced by a compact hypersurface $F^l \subset E^{l+1}$ of positive sectional curvature. It follows from the conditions on the curvature that the hypersurface F^l is convex. It bounds a domain Ω^{l+1} diffeomorphic to a ball, and the hypersurface F^l is diffeomorphic to a sphere.

12. Find all (irreducible) isometric immersions of multidimensional metrics (of dimension exceeding two) of constant positive sectional curvature into a Euclidean space (and other spaces of constant curvature) such that all intrinsic motions

are induced by motions of the enveloping space. This fact is connected with the Do Carmo–Wallach theorem [61] on minimal isometric immersion of spheres into spheres (§3.3.3). It is probable that in the general case these immersions are not exhausted by the standard immersions for which the coordinate functions are eigenfunctions corresponding to the same eigenvalue of the Laplace operator on the sphere. The problem is to distinguish natural geometric conditions under which the immersion is standard.

13. For hypersurfaces, Graves generalized the Hartman–Nirenberg theorem for isometric immersions of the pseudo-Euclidean (Lorentz) space E^l_1 into E^{l+1}_1 (Theorem 4.1.1).

Prove an analogue of Theorem 3.1.1 for an isometric immersion of the pseudo-Euclidean space E^l_s into E^{l+p}_s. A partial result in this direction is given by Theorem 4.1.2.

14. Let H^l_s be a geodesically complete pseudo-Riemannian manifold of constant negative curvature. If $s \neq 0, 1, 3, 7$, then the manifold H^l_s cannot be isometrically immersed into the pseudo-Euclidean space E^{2l-1}_s (Theorem 4.2.5).

If $s = 0$, then $H^l_0 = H^l$, and this is Problem 8.

The same negative answer is conjectured for $s = 1, 3, 7$ as well. For $s = 3, 7$ one can obtain a solution from topological obstructions and for $s = 1$ from metric obstructions, as in the case $s = 0$.

15. Sabitov proved that any locally Euclidean metric defined on a disc can be isometrically embedded in E^3 (Theorem 2.3.3). It is natural to pose the analogous question for flat metrics given in an n-dimensional ball for $n \geqslant 3$ (Gromov [79]).

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