1) $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, bilinear

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - f(x_0,k) - f(h,y_0)}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\|(h,k)\|} = 0.$$

$$Df_{(x_0,y_0)}(h,k) = f(x_0,k) + f(h,y_0).$$

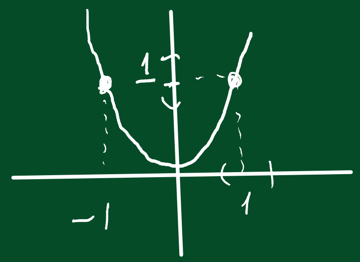
$$|f(x,y)| = \left|\sum_{i,j} x_i a_{ij} y_j\right| \leqslant M \cdot \left|\sum_{i,j} x_i y_j\right| \leqslant M \cdot c^2 \|x\| \cdot \|y\| \leqslant M c^2 \|(x,y)\|$$

Se já sabemos que f é difvel $Df(x_0,y_0)(h,k) = \frac{d}{dt}\Big|_{t=0} f((x_0,y_0)+t(h,k)) = f(x_0,k) + f(h,y_0)$.

$f : (\mathbb{R}^n)^k \to \mathbb{R}^m$ . . .

2) a) $f(X) = X^2$ é C^∞ $\qquad \mathbb{R}^{n^2} \longleftrightarrow M_n(\mathbb{R})$

↳ tem entradas que são somas e produtos das entradas de X.



b) $f(x) = X^2$ é "bilinear", $f(x) = B(x,x)$, $B(x,y) = XY$

$T(X) = (X,X)$

$$f(x) = B(T(x)) \Rightarrow Df_{X_0}(h) = (DB_{(X_0,X_0)} \circ DT_{X_0})(h)$$
$$= DB_{(X_0,X_0)}(h,h) = B(X_0,h) + B(h,X_0)$$
$$= X_0 h + h \cdot X_0.$$

c) $Y_0 = Id = f(X_0)$, $X_0 = Id$

$Df_{Id}(h) = 2 \cdot h$ é isomorfismo, pelo T.F.Inv, existem $V \subset M_n(\mathbb{R})$, $Y_0 \in V$ e $U \subset M_n(\mathbb{R})$, $X_0 \in U$ tq

$f: U \to V$ é difeo. Logo para cada $Y \in V$, $\exists\, X \in U$ tq $f(X) = Y$, $X^2 = Y$.

3) a) $SL(3) = \{A \in M_3(\mathbb{R}) : \det A = 1\}$ $\quad f : \mathbb{R}^9 = \mathbb{R}^8 \times \mathbb{R} \to \mathbb{R}$

$A = (a_1, a_2, a_3) \in \mathbb{R}^9 \approx M_3(\mathbb{R})$

$f(X) = \det X$ é C^∞ e multilinear

$$Df_A(h) = \begin{vmatrix} a_1 \\ a_2 \\ h_3 \end{vmatrix} + \begin{vmatrix} a_1 \\ h_2 \\ a_3 \end{vmatrix} + \begin{vmatrix} h_1 \\ a_2 \\ a_3 \end{vmatrix} \Longrightarrow Df_A(A) = 3, \forall A \in SL(3).$$

Pelo T.F.Imp, um das entradas de f é função das demais numa vizinhança de cada $A \in SL(3)$.

b) $\gamma : \mathbb{R} \to SL(3)$, $A + th$ $\Rightarrow \det(\gamma(t)) = 1 \Rightarrow D\det_{\gamma(t)}(\gamma'(t)) = 0 \Rightarrow \gamma'(t) \in \operatorname{Ker} D\det_{\gamma(t)}$.

$D\det_{\gamma(t)} : \mathbb{R}^9 \to \mathbb{R} \Rightarrow \dim \operatorname{Ker} D\det_{\gamma(t)} = 8.$

c) $\gamma(t) = Id + th \Rightarrow D\det_I(h) = \left.\dfrac{d}{dt}\right|_{t=0} \det(I + th) = \operatorname{tr} h \Rightarrow h \in \operatorname{Ker} D\det_I \Leftrightarrow \operatorname{tr} h = 0$

$\det(A + tI) = t^n + \operatorname{tr} A\, t^{n-1} + \dots + \quad + \det A$

$\Downarrow$

$\det(A + \frac{1}{t} I) = t^{-n} + \operatorname{tr} A \cdot t^{1-n} + \dots + \det A$

$\det(tA + I) = 1 + \operatorname{tr} A \cdot t + \dots + \det A \cdot t^n$

$\Downarrow$

$\left.\dfrac{d}{dt}\right|_{t=0} \det(tA + I) = \operatorname{tr} A$

Podemos provar que $D\det_A(h) = \det(A) \cdot \operatorname{tr}(A^{-1} \cdot h)$.

4) Análogo ao anterior, pois posto $A = 1 \iff \det A = 0$ e $A \neq 0$ $\quad A \in M_2(\mathbb{R})$.

$\det : M_2(\mathbb{R}) \to \mathbb{R}$

Se $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $D\det_A \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} + \begin{vmatrix} d & -c \\ c & d \end{vmatrix} = a^2 + b^2 + c^2 + d^2 > 0$, pois $A \neq 0$

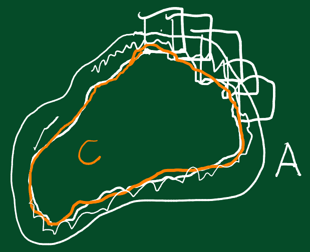
TF Imp $\Rightarrow$ dá uma entrada em termos das demais

$A + th = \begin{bmatrix} 0 + th_1 & 0 + th_2 \\ 0 + th_3 & 1 + th_4 \end{bmatrix} \Rightarrow \det(A + th) = (h_1 h_4 - h_3 h_2) t^2 + th_1$

Esp. tang é Ker $D\det_A$: tem dimensão 3.

Se $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\gamma(t) = A + t \cdot h \Rightarrow D\det_A(h) = \left.\dfrac{d}{dt}\right|_{t=0} \det(A + th) = h_1$

$\operatorname{Ker} D\det_A = \begin{bmatrix} 0 & h_2 \\ h_3 & h_4 \end{bmatrix}$

5) 

A é J-mensurável $\Rightarrow$ A é limitado e $m(\partial A)=0$

$\hookrightarrow$ dado $\varepsilon>0$, $\exists \overset{\text{ret. abertos}}{U_i}$, $i\in\mathbb{N}$, tq $\partial A \subset \bigcup U_i$ e $\sum_{i\in\mathbb{N}} m(U_i) < \varepsilon$

∂A é fechado e limitado $\Rightarrow \partial A$ é compacto

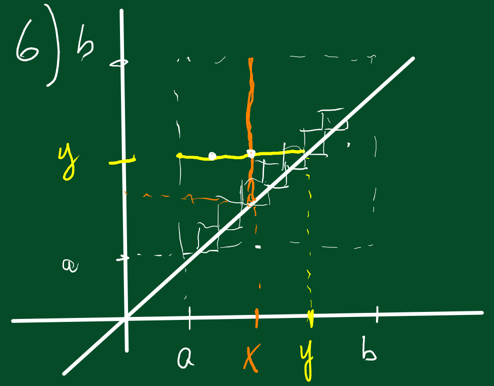
$\Rightarrow \{U_i\}_{i=1}^{k}$ tq $\partial A \subset \bigcup_{i=1}^{k} U_i$, $\sum_{i=1}^{k} m(U_i) < \varepsilon$

$C = \bar{A} \setminus \underbrace{\bigcup_{i=1}^{k} U_i}_{U}$ é fechado e limitado (compacto)

$\partial C \subset \underbrace{\partial \bar{A}}_{\text{tem med nula}} \cup \underbrace{\partial U}_{\text{tem med. nula}}$ tem medida nula

$A\setminus C \subset \bar{U} \Rightarrow \int_{A\setminus C} 1 \leq \int_{\bar{U}} 1 = \mathrm{vol}(U) \leq \ < \varepsilon.$

6)



$$\int_a^b \int_a^y f(x,y)\,dx\,dy$$

$$\int_a^b \int_x^b f(x,y)\,dy\,dx$$

$$\{(x,y)\in\mathbb{R}^2 : a\leq y\leq b\ ,\ a\leq x\leq y\} = \{(x,y)\in\mathbb{R}^2 : a\leq x\leq b,\ x\leq y\leq b\}$$

$$h_x(y) = f(x,y)\cdot\chi_{[x,b]}$$

$$g_y(x) = f(x,y)\cdot\chi_{[a,y]}\ ,\ g_y : [a,b]\to\mathbb{R}$$

$$\int_a^b\int_a^y f(x,y)\,dx\,dy = \int_a^b\int_a^b g_y(x)\,dx\,dy = \int_a^b\int_a^b g_y(x)\,dy\,dx = \int_a^b\int_a^b h_x(y)\,dy\,dx = \int_a^b\int_x^b f(x,y)\,dy\,dx.$$

7) $F:[c,d]\to\mathbb{R}$, $F(y)=\int_a^b f(x,y)\,dx$, f contínua

$$F'(y)=\frac{d}{dy}\int_a^b f(x,y)\,dx=\int_a^b\frac{\partial f}{\partial y}(x,y)\,dx.$$

Escreva $F(y)=\int_a^b\left(\int_c^y\frac{\partial f}{\partial t}(x,t)+f(x,c)\,dt\right)dx$

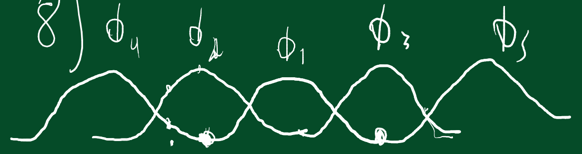
$$=\underbrace{\int_a^b\int_c^y\frac{\partial f}{\partial t}(x,t)\,dt\,dx}_{\displaystyle \| \atop \displaystyle \int_c^y\int_a^b\frac{\partial f}{\partial t}(x,t)\,dx\,dt}+\underbrace{\int_a^b\int_c^y f(x,c)\,dt\,dx}_{\bar{n}\text{ depende de } y}$$

$$\Rightarrow F'(y)=\int\frac{\partial f}{\partial y}(x,y)\,dx.$$

$$\int_c^d F(y)\,dy=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

$$f(x,y)=f_x(y)=\int_c^y\frac{d f_x}{dt}(t)\,dt-f(x,c)$$

8)



de clase $\mathcal{C}^1$

$$\operatorname{supp} \phi_K = \begin{cases} \left[\frac{K-3}{2}\pi, \frac{K+1}{2}\pi \right], & K \text{ ímpar} \\ \left[-\frac{K-2}{2}\pi, \frac{2-K}{2}\pi \right], & K \text{ par} \end{cases}$$