LECTURES ON MODEL-THEORETIC FORCING

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ABSTRACT. We present the finite and the infinite forcing of Abraham Robinson.

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Foreword

These are extended notes for a short course on Forcing in Model Theory for the *Cantor Meets Robinson* conference held in São Paulo and Campinas, state of São Paulo, Brazil, on December, 2018.

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1. Model Theoretic Preliminaries

We assume familiarity with the basic results of Predicate Calculus: Completeness, Compactness, Löwenheim-Skolem. You can read about these in your preferred Model Theory book.

We recall only the basics of infinitary logic and some specific concepts and results from Model Theory which are akin to model theoretic forcing.

2. Infinitary Language

We also work with a logic which admits formulas of infinite length. The one considered here is named $L_{\omega_1\omega}$ and extends the first order logic (also named $L_{\omega\omega}$).

Definition 2.1. The language of first order logic is extended with two new logical symbols, $\bigwedge$ and $\bigvee$. The construction rules for these symbols are: given a non empty and countable (finite or denumerable infinite) set of formulas Φ, $\bigwedge\Phi$ and $\bigvee\Phi$ are formulas. If the set Φ carries an enumeration $\Phi = \{\phi_n : n \in \omega\}$, the we may write instead $\bigvee_{n<\omega}\phi_n$ and $\bigwedge_{n<\omega}\phi_n$. (This is important whenever we need to talk about recursive relations and functions.)

Structures are the same as before, with the satisfaction relation extended by $\mathcal{M} \models \bigwedge\Phi[s]$ if and only if, for each $\phi \in \Phi$ $\mathcal{M} \models \phi[s]$, and $\mathcal{M} \models \bigvee\Phi[s]$ if and only if, for some $\phi \in \Phi$ $\mathcal{M} \models \phi[s]$.

The notions of free and bound variables for $L_{\omega_1\omega}$ is *mutatis mutandis* the same as for first order formulas. The only difference is the cases $FV(\bigwedge_{n\in\mathbb{N}}\psi_n) = FV(\bigvee_{n\in\mathbb{N}}\psi_n) = \bigcup_{n\in\mathbb{N}} FV(\psi_n)$. Now the function FV has values countable sets of variables.

The full language $L_{\omega_1\omega}$ has always uncountably many formulas and we may be interested in only a *well behaved* subset of such formulas (in particular, a countable subset). So we define such *fragments*, but start with the necessary concept of sub-formulas.

Definition 2.2 (Sub-formulas)**.** We define recursively the function SF on formulas of $L_{\omega_1\omega}$ with values countable sets of formulas as follows.

If ϕ is an atomic formula then $SF(\phi) = \{\phi\}$. Suppose that we have defined $SF(\phi)$ and $SF(\psi)$. We define $SF(\neg\phi) = \{\neg\phi\} \cup SF(\phi)$ and $SF(\phi \circ \psi) = \{\phi \circ \psi\} \cup SF(\phi) \cup SF(\psi)$ where $\circ$ stands for any of the symbols $\wedge$, $\vee$, $\to$ and $\leftrightarrow$. We define $SF(\exists x_i\psi) = \{\exists x_i\psi\} \cup SF(\psi)$ and $SF(\forall x_i\psi) = \{\forall x_i\psi\} \cup SF(\psi)$. Now we suppose that we have defined $SF(\phi_n)$ for all formulas of the sequence $(\phi_n)_{n\in\mathbb{N}}$. We define $SF(\bigwedge_{n\in\mathbb{N}}\phi_n) = \{\bigwedge_{n\in\mathbb{N}}\phi_n\} \cup \bigcup_{n\in\mathbb{N}} SF(\phi_n)$ and $SF(\bigvee_{n\in\mathbb{N}}\phi_n) = \{\bigvee_{n\in\mathbb{N}}\phi_n\} \cup \bigcup_{n\in\mathbb{N}} SF(\phi_n)$.

We say that ϕ is a *sub-formula* of ψ if $\phi \in SF(\psi)$.

Definition 2.3 (Admissible Fragment of $L_{\omega_1\omega}$)**.** An *admissible fragment*, or just a *fragment*, of $L_{\omega_1\omega}$ is a set of formulas L_A containing the first order formulas, closed under quantification (if $\phi \in L_A$ then $\exists x_i\phi \in L_A$ and $\forall x_i\phi \in L_A$), sub-formulas (if $\phi \in L_A$ then $SF(\phi) \subseteq L_A$), and if $\phi(x) \in L_A$ and τ is a term then $\phi(\tau) \in L_A$.

Remark 2.4. The first order logic is an admissible fragment of $L_{\omega_1\omega}$, the smallest one.

Remark 2.5 (Larger Infinitary Languages). We can sometimes work with infinitary languages with uncountable conjunctions of formulas. For these we assume that the set of variables is indexed by a limit ordinal of cardinality κ, for some regular cardinal $\kappa > \aleph_0$, but still finitely many quantifiers (we do not consider here languages with infinite strings of quantifiers). The terms and formulas are defined in the usual manner, but we admit formulas of the type $\bigvee\Phi$ and $\bigwedge\Phi$, where the cardinality of the set of formulas Φ does not exceed κ.

If we do not impose the restriction on the cardinality of Φ we obtain the logic denoted $L_{\infty\omega}$. The satisfaction relation is defined accordingly.

3. Preservation Results

In this section we collect some preservation results which allows us to prove some syntactic characterizations of *generic models* and *forcing*.

Just for the record:

Definition 3.1 (Theory)**.** A *theory* is a consistent set of sentences. No completeness assumption is added. We say that the theory T is *axiomatized* by the theory Γ if both have the same logical consequences (or, in the case of first order, the same theorems).

Definition 3.2 (Prenex Form)**.** A first order formula ϕ is in *prenex form* if it is of the type $Q_1x_{i_1}\dots Q_nx_{i_n}\psi$, where each Q_i is a quantifier "$\exists$" or "$\forall$" and ψ is a quantifier-free formula. If L is a signature, we denote by $\exists_1(L)$ (or just $\exists_1$ if L is understood in the context) the set of formulas logically equivalent to one in prenex form which contains only existential quantifiers, $\forall_1(L)$ (or just $\forall_1$) the set of formulas logically equivalent to one in prenex form which contains only universal quantifiers. We define recursively $\exists_{n+1}(L)$ and $\forall_{n+1}(L)$ the sets of formulas logically equivalent to one of the form $Q_1 x_{i_1} \dots Q_n x_{i_n} \psi$, where $\psi \in \forall_n(L)$ or $\psi \in \exists_n(L)$, respectively.

Observe that $L_{\infty\omega}$-formulas may not admit a prenex form due to the fact that we cannot bring forward the quantifiers in infinite conjunctions or disjunctions of a set of formulas with an unbounded quantity of quantified variables, as we can see in the sentence $\bigwedge_{n<\omega} \phi_n$, where ϕ_n says that there are at least $n+1$ distinct elements.

We put the consistency assumption on a theory T to avoid unnecessary extra hypotheses in the results we deal with.

Remark 3.3 (Chain of Structures). A *chain of structures* is a sequence $(M_\alpha)_{\alpha<\lambda}$ os L-structures such that for each pair $\alpha < \beta < \lambda$, M_α is a substructure of M_β.

Firstly we prove Tarski's elementary chain theorem.

Theorem 3.4. Let $(M_\alpha)_{\alpha<\lambda}$ be an elementary chain of structures, that is, if $\alpha < \beta$, then $M_\alpha \prec M_\beta$. Let $M = \bigcup_{\alpha<\lambda} M_\alpha$ turned naturally into an L-structure. In this setting, for each $\alpha < \lambda$, $M_\alpha \prec M$.

Proof. We prove by induction on the complexity of formulas that for $\bar{a}$ in M_α, $M \models \phi(\bar{a})$ if, and only if, $M_\alpha \models \phi(\bar{a})$.

The basic step (atomic formulas) and the induction steps for the propositional connectives are easy and left as exercises.

We treat here the case where $\phi(\bar{x})$ is $\exists y \psi(\bar{x}, y)$, and assume the induction hypothesis that the statement is true for ψ.

If $M \models \phi(\bar{a})$, then there is some $b \in M$ such that $M \models \psi(\bar{a}, b)$. By the inductions hypothesis, there is some $\xi \geq \alpha$ for which $M_\xi \models \psi(\bar{a}, b)$ (the index ξ is any one such that $\bar{a}$ and b are in M_ξ). By definition, $M_\xi \models \exists y \psi(\bar{a})$ and because $M_\alpha \preceq M_\xi$, $M_\alpha \models \phi(\bar{a})$. Analogous argumentation shows that if $M_\alpha \models \phi(\bar{a})$, then $M \models \phi(\bar{a})$.

The equivalence between $\forall y \psi(\bar{x}, y)$ and $\neg[\exists y \neg\phi(\bar{x}, y)]$ releases us the proof of this case. ☐

For the next three theorems we prove firstly the following lemma which contains the structure of their proofs.

Lemma 3.5. Let Δ be a set of first order sentences closed under finite disjunctions. The following are equivalent.

(1) The theory T has a set $\Gamma \subseteq \Delta$ of axioms.

(2) For all $M \models T$ and M' an L-structure, if for all $\delta \in \Delta$, $M \models \delta$ implies $M' \models \delta$, then the L-structure M' must be also a model of T.

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

Assume (2) and let $\Gamma = \{\delta \in \Delta : T \models \delta\}$. It is clear that $T \models \Gamma$. We need to show that $\Gamma \models T$.

Let $M \models \Gamma$ be an arbitrary model and let $\Sigma = \{\neg\delta : M \models \neg\delta, \delta \in \Delta\}$. Observe that if $\neg\delta \in \Sigma$, then $\delta \notin \Gamma$ (this is the fact to be contradicted below). We claim that $T \cup \Sigma$ is consistent, because otherwise there would be $\delta_1, \ldots, \delta_n \in \Delta$ such that $T \vdash \neg(\neg\delta_1 \wedge \cdots \wedge \delta_n)$ (this is Compactness), that is, $T \vdash (\delta_1 \vee \cdots \vee \delta_n)$. Because Δ is closed under finite disjunctions the sentence $(\delta_1 \vee \cdots \vee \delta_n)$ is in Γ and so $M \models (\delta_1 \vee \cdots \vee \delta_n)$, a contradiction.

If $M' \models T \cup \Sigma$ then (2) implies that $M \models T$. Therefore $\Gamma \models T$. $\square$

We now apply this to the sets $\forall_1$, $\forall_2$ and of *positive formulas* (those built up from atomic formulas, $\vee$, $\wedge$, $\exists$, $\forall$; we do not use negation in any form, and we recall that implication has a built in negation). Any one of those sets are closed under finite disjunctions.

Theorem 3.6. The class of models of T is closed under substructures if, and only if, T has a $\forall_1$ set of axioms.

Proof. The right to left implication is simple and left to the reader.

Let Δ be the set of all sentences logically equivalent to $\forall_1$ sentences.

Assume now that the class of models of T is closed under substructures and let $\Gamma = \{\delta \in \Delta : T \models \delta\}$. Our objective is to prove that $\Gamma \models T$. Let $M \models T$ any model and M' a model of the universal sentences true in M. Let $T' = T \cup \Delta(M')$, where $\Delta(M')$ is the diagram of M'. Compactness, plus the fact that the existential sentences true in M' must also be true in M, imply that T' is consistent. Any model $M'' \models T'$ has an isomorphic copy of M' as a substructure. Therefore $M' \models T$ because the class of models of T are closed under substructures. Lemma 3.5 ends our proof. $\square$

Example 3.7. A typical example of a $\forall_1$ theory is the Theory of Rings. A substructure of a ring is a sub-ring which is also a ring *per se*. The signature contains the constant symbols “0” and “1”, the unary operation symbol “$-$” and the binary operation symbols “$+$” and “$\cdot$”. A list of axioms is $\forall x(x + 0 = x)$, $\forall x \forall y(x + y = y + x)$,

$\forall x \forall y \forall z[(x+y)+z = z+(y+z)]$, $\forall x[x+(-x)=0]$, $\forall x[x \cdot 0 = 0]$, $\forall x[x \cdot 1 = x]$, $\forall x[1 \cdot x = x]$, $\forall x \forall y \forall z[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ and $\forall x \forall y \forall z[x \cdot (y+z) = (x \cdot y) + (x \cdot z)]$.

Theorem 3.8. The class of models of T is closed under union of chains of its models if, and only if, T has a set of $\forall_2$ axioms.

Proof. The easy part, the right side implies the left side, is left as an exercise.

Now we assume that the class of models of T is closed under union of chains of its models and prove that T has a set of $\forall_2$ axioms. We use again Lemma 3.5, with Δ as the set of all sentences logically equivalent to $\forall_2$ sentences. Let $M \models T$ and $M' \models \phi$ for all $\phi \in \Delta$ such that $M \models \phi$. We want to show that $M' \models T$.

In order to do this we construct an increasing chain of structures $M' = M'_0 \subset M_1 \subset M'_1 \subset M_2 \subset \dots$ with $M_k \equiv M$ and $M'_k \prec M'_{k+1}$ as follows.

Suppose we have constructed the sequence up to the level $n \geq 0$ with the desired properties.

Let $(M'_n, m)_{m \in M'_n}$ be the expansion of the model M'_n by interpreting the new constant symbols c_m, $m \in M'_n$. We indicate this model again as M'_n. Let T_1 be the L-theory of M and $T_2 = \{\phi \in \forall_1(L(M'_n)) : M'_n \models \phi\}$. By existentially quantifying out the new constants we deduce that $T_1 \cup T_2$ is consistent, because of the induction hypothesis on the pair of models $M_n \equiv M$ and $M'_n \succ M'$. Let $M_{k+1} \models T_1 \cup T_2$. This model is elementary equivalent to M and contains an isomorphic copy of M'_k and so we can assume that $M'_k \subset M_{k+1}$. Because of T_2 being what it is every universal $L(M'_k)$-sentence true in M'_k must be true in M_{k+1} what implies that every existential $L(M'_k)$-sentence true in M_{k+1} is also true in M'_k.

We now extend this model to an elementary extension $M'_{k+1} \succ M'_k$. The diagram $\Delta(M_{k+1})$ is consistent with $Th_{L(M'_k)}(M_k)$ by Compactness and the last observation of the previous paragraph. Select a model $M'_{k+1} \models \Delta(M_{k+1}) \cup Th_{L(M'_k)}(M_k)$. This model has an isomorphic copy of M_{k+1}, which itself has a copy of M'_k, and is elementary equivalent to M'_k (in the language $L(M'_k)$) and so $M'_k \prec M'_{k+1}$ as desired. □

Example 3.9. One typical example here is the Theory of Fields in the signature of the Theory of Rings. Add the following axiom to the axioms for rings, $\forall x[(x=0) \vee \exists y(x \cdot y = 1)]$.

Example 3.10. Another important example is the theory of groups in the signature containig one constant symbol "e" and one binary operation symbol "$\cdot$". One set of axioms contains $\forall x(x \cdot e = x)$, $\forall x \forall y \forall z[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$, $\forall x \exists y[(x \cdot y = e) \wedge (y \cdot x = e)]$.

Observe that substructures of a group may be just a semigroup (closed under multiplication but not necessarily under inverse). If one introduces an operation symbol for the inverse then there will be a set of universal axioms.

4. Existentially Closed Structures and Model Completeness

Some important algebraic structures have some closure property. For instance, algebraically closed fields, or groups, real closed fields, differentially closed fields. We present here two such closely related notions, the existential closeness and model completeness. These are important notions to the idea and application of forcing.

Definition 4.1 (Diagram of an L- Structure)**.** Let M be an L-structure. We expand the signature L to the signature $L(M)$ by joining a new constant symbol c_m, for each $m \in M$. The *diagram* of M is the set $\Delta(M)$ of *basic sentence* (an atomic sentence or the negation of an atomic sentence) that are satisfied in M.

Definition 4.2 (Existentially Closed Structures)**.** An L structure M is *existentially closed* if for all L structures $M' \supseteq M$, all $\phi(\bar{x}) \in \exists_1(L)$, and all $\bar{A}$ in M, if $M' \models \phi(\bar{a})$, then $M \models \phi(\bar{a})$. We denote this as $M \prec_{\exists_1} M'$. Let T be an L-theory. We say that the model $M \models T$ is *existentially closed over* T if forall $M' \supseteq M$, such that $M' \models T$, $M \prec_1 M'$.

The following result is a useful characterization of existentially closed structures, due to Harold Simmons, [22, Theorem 2.1, p. 297].

Theorem 4.3. Let M be a model of a first order theory T. The following are equivalent:

(1) The structure M is existentially closed over T.
(2) If $M \models \phi(\bar{a})$, for some $\phi \in \forall_1$, then $T \cup \Delta(M) \vdash \phi(\bar{c}_{\bar{a}})$, where $\bar{a} = (a_1, \ldots, a_n) \in M^n$ and $\bar{c}_{\bar{a}} = (c_{a_1}, \ldots, c_{a_n})$ is the tuple of corresponding new constant symbols.
(3) If $M \models \phi(\bar{a})$, for some $\phi \in \forall_1$, then $M \models \theta(\bar{a})$, for some $\theta \in \exists_1$ such that $T \vdash \forall \bar{x}[\theta(\bar{x}) \rightarrow \phi(\bar{x})]$.

Proof. We prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Suppose that $M \models \phi(\bar{a})$, for some $\phi \in \forall_1$, and that M is existentially closed. Any model $M' \models T$ containing M can be expanded naturally to a model of $T \cup \Delta(M)$. Since M is existentially closed, for all $M' \models T \cup \Delta(M)$, $M' \models \phi(\bar{a})$. This means that $\phi(\bar{c}_{\bar{a}})$ is a logical consequence of $T \cup \Delta(M)$ and, by the Completeness Theorem, $T \cup \Delta(M) \vdash \phi(\bar{c}_{\bar{a}})$.

(2) $\Rightarrow$ (3): Suppose that $M \models \phi(\bar{a})$, for some $\phi \in \forall_1$. By (2), $T \cup \Delta \vdash \phi(\bar{c}_{\bar{a}})$, which means that there exists a quantifier free L-formula $\delta(\bar{x}, \bar{y})$ and tuple $\bar{a}'$ in M, such that $T \vdash \delta(\bar{c}_{\bar{a}}, \bar{c}_{\bar{a'}}) \to \phi(\bar{c}_{\bar{a}})$ and, therefore, $T \vdash \exists \bar{y}[\delta(\bar{c}_{\bar{a}}, \bar{y}) \to \phi(v)]$. Since the constants $\bar{c}_{\bar{a}}$ do not occur in the sentences of T, $T \vdash \forall \bar{x}[\theta(\bar{x}) \to \phi(\bar{x})]$, where θ is the existential formula $\exists \bar{y} \delta(\bar{x}, \bar{y})$.

(3) $\Rightarrow$ (1): Assume (3) but suppose that M is not existentially closed. Then there is some $\phi(\bar{x}) \in \forall_1$, some $\bar{a}$ in M, and some $M' \models T$, with $M \subset M'$, such that $M' \models \neg\phi(\bar{a})$ and $M \models \phi(\bar{a})$. Let $\theta(\bar{x}) \in \exists_1$, such that $T \vdash \forall \bar{x}[\theta(\bar{x}) \to \phi(\bar{x})]$ and $M \models \theta(\bar{a})$. Since θ is existential, it is preserved upwards, so $M' \models \theta(\bar{a})$ and so $M' \models \phi(\bar{a})$ because $M' \models \forall \bar{x}[\theta(\bar{x}) \to \phi(\bar{x})]$. This contradiction proves the statement (1). $\square$

Another closely related notion of completeness is that of model completeness.

Definition 4.4 (Model Completeness)**.** A first order L-theory T is *model complete* if for any $M \models T$, $T \cup \Delta(M)$ is a complete $L(M)$-theory.

There are other equivalent formulations of this concept.

Theorem 4.5. Let T be a first order theory. These statements are equivalent:

(1) The theory T is model complete.
(2) Every embedding $M_1 \to M_2$ of L-structures, where $M_1, M_2 \models T$ is elementary.
(3) Every model $M \models T$ is existentially complete.
(4) For each L-formula $\phi(\bar{x})$, whose free variables are among the list $\bar{x}$, there is a formula $\phi(\bar{x}) \in \forall_1(L)$ with the same free variables, such that $T \vdash \forall \bar{x}(\phi \leftrightarrow \psi)$.
(5) For each L-formula $\phi(\bar{x})$, whose free variables are among the list $\bar{x}$, there is a formula $\phi(\bar{x}) \in \exists_1(L)$ with the same free variables, such that $T \vdash \forall \bar{x}(\phi \leftrightarrow \psi)$.

Proof. We prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Assume that T is model complete and let $M_1, M_2 \models T$, with M_1 a substructure of M_2. We can expand M_1 and M_2 to $L(M_1)$-structures, both being models of $T \cup \Delta(M_1)$. Let $\phi(\bar{x})$ be an L-formula and $\bar{a}$ in M_1, such that $M_2 \models \phi(\bar{a})$. The theory $T \cup \Delta(M_1)$ is complete by hypothesis and so $T \cup \Delta(M_1) \vdash \phi(\bar{c}_{\bar{a}})$. Therefore $M_1 \models \phi(\bar{a})$. This means that M_1 is an elementary substructure of M_2.

(2) $\Rightarrow$ (3): This is immediate.

(3) $\Rightarrow$ (4): By the application of induction on the complexity of formulas we can reduce this implication to the case of an existential formula.

Let $\phi \in \exists_1(L)$. The introduction of finitely many new constants to the signature allows us to assume that ϕ is a sentence. Let Γ be the set of all sentences $\gamma \in \forall_1$ such that $T \vdash (\phi \rightarrow \gamma)$. We can assume that $T \cup \{\phi\}$ is consistent, because otherwise we can choose any inconsistent $\gamma \in \forall_1$ and $T \vdash (\phi \leftrightarrow \gamma$. Let $M \models T \cup \Gamma$. The set of sentences $T \cup \{\phi\} \cup \Delta(M)$ is consistent because for each finite conjunction $\theta(\bar{a})$ of formulas, $\forall \bar{x} \theta(\bar{x})$ is false in M and so it does not belong to Γ. There is a model $M' \models T \cup \{\phi\} \cup \Gamma$, such that $M \subseteq M'$. By (3), M is existentially closed and so $M \models \phi$. We have then proved that ϕ is a logical consequence of $T \cup \Gamma$, and by compactness, there is a finite $\Gamma_0 \subset \Gamma$, such that $T \vdash \bigwedge \Gamma_0 \rightarrow \phi$. By the definition of Γ and logical tricks, we obtain the desired $\gamma \in \forall_1$ such that $T \vdash (\phi \leftrightarrow \gamma$.

(4) $\Leftrightarrow$ (5): This is a simple logical exercise.

(4) $\Rightarrow$ (1): Let $M, M' \models T$ with M a substructure of M', and $\phi(\bar{x})$ an L-formula such that $M' \models \phi(\bar{a})$ for some $\bar{a}$ in M. Let $\psi(\bar{x}) \in \forall_1$ be such that $T \models \phi \leftrightarrow \psi$, and we write ψ as $\forall \bar{y} \theta(\bar{x}, \bar{y})$. This implies that $M \models \forall \bar{y} \theta(\bar{a}, \bar{y})$ because universal formulas are preserved downwards. Therefore $T \cup \Delta(M) \vdash \forall \bar{y} \theta(\bar{c}_{\bar{a}}, \bar{y})$, because M' is arbitrary. From this we can conclude that $T \cup \Delta(M) \vdash \phi(\bar{c}_{\bar{a}})$, that is, T is model complete. □

Example 4.6. Model complete theories play an important role in Algebra. Some known model complete theories (in their natural signatures) are: (1) dense linear order without endpoints, (2) algebraically closed fields, (3) real closed fields, (4) p-adic fields, (5) atom-less Boolean algebras, (6) the additive group of integers with 1 as distinguished element, (7) the non negative integers with the constant 0 and the successor operator, (8) divisible torsion free abelian groups, (9) divisible ordered groups, (10) the field of real numbers with the exponential function,

(11) the field of real numbers with restricted real analytic functions, (12) the field of real numbers with restricted real analytic functions and the unrestricted exponential functions, (13) differentially closed fields of characteristic zero, and so on.

The proofs of model completeness of these theories can be lengthy and involve lots of knowledge from other areas of Mathematics.

Example 4.7. We know that any field can be embedded in an algebraically closed field. So the role played by the latter in the class of all fields suggests another concept.

Definition 4.8 (Companion Theories)**.** A first order L-theory T^* is a *model companion* of the L-theory T if both have the same universal consequences (we denote $T_\forall = T^*_\forall$) and T^* is model complete. If $T \subset T^*$, then T^* is a *model completion* of T.

Remark 4.9. Given a first order theory T, $M \models T_\forall$ if, and only if, there is $M' \models T$ such that M is a substructure of M'. Indeed, given a model $M \models T_\forall$, let Σ be the set of existential L sentences true in M. The set $T \cup \Sigma$ is consistent because, otherwise, the usual argumentation we obtain an existential sentence $\phi \in \Sigma$ such that $T \vdash \neg\phi$, and so $M \models \neg\phi$. This contradicts $M \models \phi$.

Lemma 4.10. A first order theory T has at most one model companion.

Proof. Let T^* and T^{**} be two model companions of T. They have the same universal consequences as T, so $T^*_\forall = T^{**}_\forall$. We build a chain of models $M_0 \subset M_1 \subset \dots$ where for each $k \geq 0$ $M_{2k} \models T^*$ and $M_{2k+1} \models T^{**}$. The odd and the even indexed sub-sequences are elementary chains (because of model completeness). The union of the chain is a model of both T^* and T^{**}. □

We see in the following chapter the notion of forcing and its relation with existential closure and model completeness.

5. Forcing Notions

We fix an admissible countable fragment L_A of $L_{\omega_1\omega}$. Let $D = \{d_n : n \in \mathbb{N}\}$ be a set of constant symbols disjoint from the ones from the original language. Let K_A be the set of all formulas of the form $\phi|_{x_{i_1}=d_{j_1},\dots,x_{i_n}=d_{j_n}}$ (the substitution of all the free instances of the variables x_{i_j} by the corresponding constant symbol d_{i_j}), for ϕ in L_A, $d_{j_1},\dots,d_{j_n} \in D$. In order to avoid such a cumbersome notation we use

$\phi(\bar{x})$ to denote the fact that the free variables of ϕ are among the ones in the (countable) tuple $\bar{x}$, and $\phi(\bar{d})$ the substitution just described.

Unless otherwise stated, T is a *theory*, that is, a (not necessarily complete) consistent set of sentences from the fragment L_A. We will be precise when the fragment is the first order logic.

Definition 5.1. Let T be a theory. A *forcing notion* for the theory T is a triple $\mathcal{P} = \langle P, \leq, f \rangle$ where

(a) $\langle P, \leq \rangle$ is a partially ordered set with a least element $0 \in P$;
(b) f is a function from P into the set of finite subsets of atomic sentences of K_A, each of which consistent with T;
(c) if $p \leq q$ then $f(p) \subset f(q)$;
(d) for terms σ and τ of K_A without variables and $p \in P$:
  (i) if $(\sigma = \tau) \in f(p)$, then there is some $q \geq p$ such that $(\tau = \sigma) \in f(q)$;
  (ii) if $(\tau = \sigma), \phi(\tau) \in f(p)$, then there is some $q \geq p$ such that $\phi(\sigma) \in f(q)$;
  (iii) for some $d \in D$, there is some $q \geq p$ such that $(\tau = d) \in f(q)$.

Remark 5.2. This definition differs from Robinson's original one, [20, § 2, p. 70], in which the conditions are finite sets of *basic sentences*. These sentences are atomic or the negation of an atomic sentence.

Definition 5.3 (Forcing Relation)**.** Let $\langle P, \leq, f \rangle$ be a forcing notion. We define the *forcing relation* $p \Vdash \phi$ (read "*p forces* ϕ"), for $p \in P$ and ϕ a K_A sentence, recursively as follows:

(a) if ϕ is an atomic sentence then $p \Vdash \phi$ if, and only if, $\phi \in f(p)$;
(b) $p \Vdash \neg\phi$ if, and only if, for no $q \geq p$, q forces ϕ;
(c) $p \Vdash (\phi \wedge \psi)$ if, and only if, $p \Vdash \phi$ and $p \Vdash \psi$;
(d) $p \Vdash \bigwedge_{n \in \mathbb{N}} \phi_n$ if, and only if, for all $n \in \mathbb{N}$ $p \Vdash \phi_n$;
(e) $p \Vdash \exists x_i \phi$ if, and only if, for some $d \in D$ $p \Vdash \phi|_{x_i = d}$.

Remark 5.4. See the exercises in the end of this chapter on how to define the forcing relation applied to the other logical symbols.

Definition 5.5 (Weak Forcing Relation)**.** We say that p weakly forces the K_A-sentence ϕ, $p \Vdash^w \phi$, if $p \Vdash \neg\neg\phi$.

Remark 5.6. As usual, we write the negation of these relations as "$\nVdash$" and "$\nVdash^w$".

Let us first prove some basic facts about the forcing relation. These results imitate Cohen's forcing

Lemma 5.7. Let $\langle P, \leq, f\rangle$ be a forcing notion and ϕ a K_A-sentence.

(a) If $p \leq q$ and $p \Vdash \phi$ then $q \Vdash \phi$.
(b) If $p \Vdash \phi$ then $p \nVdash \neg\phi$, and if $p \Vdash \neg\phi$ then $p \nVdash \phi$.
(c) For $p \in P$, $p \Vdash^w \phi$ if, and only if, for each $q \geq p$ there is some $r \geq q$ such that $r \Vdash \phi$.
(d) If $p \Vdash \phi$ then $p \Vdash^w \phi$.
(e) For $p \in P$, $p \Vdash \neg\phi$ if, and only if, $p \Vdash^w \neg\phi$.

Proof. The proof is an easy consequence of the definitions.

(a) This item is proven by induction on the complexity of the sentence ϕ. If $p \Vdash \phi$ and ϕ is atomic then $\phi \in f(p)$ and since f is non decreasing if $q \geq p$, then $\phi \in f(q)$, which means that $q \Vdash \phi$. The induction steps follow directly from the definitions.

(b) If $p \Vdash \phi$ then the definition of the forcing relation implies that $p \nVdash \neg\phi$. If $p \Vdash \neg\phi$, then again the definition implies that $p \nVdash \phi$.

(c) If for no $q \geq p$ $q \Vdash \phi$, then $p \Vdash \neg\phi$ and, by the item (b), $p \nVdash \neg\neg\phi$. Conversely, if $p \nVdash \neg\neg\phi$, then for some $q \geq p$ $q \Vdash \neg\phi$ and, therefore, for no $r \geq q$ $r \Vdash \phi$.

(d) If $p \Vdash \neg\phi$ then for all $q \geq p$ $q \Vdash \neg\phi$, by the item (a). By the item (b), for no $q \geq p$ $q \Vdash \neg\neg\phi$ and, therefore, $p \Vdash \neg\neg\neg\phi$. Conversely, if for no $q \geq p$ $q \Vdash \neg\phi$ then $p \Vdash \neg\neg\phi$. By the item (b), $p \nVdash \neg\neg\neg\phi$. □

Definition 5.8. Given the forcing notion $\langle P, \leq, f\rangle$ and a theory T, we denote T^f as the set of K_A sentences weakly forced by some $p \in P$.

6. Generic Models

We show in this section how to construct a model from a forcing notion.

Definition 6.1 (Generic Sets)**.** Let $\mathcal{P} = \langle P, \leq, f\rangle$ be a forcing notion. A subset $G \subset P$ is *generic* if, and only if,

(a) if $p \in G$ and $q \leq p$, then $q \in G$;
(b) for all $p, q \in G$, there is some $r \in G$ with $p \leq r$ and $q \leq r$;
(c) for each K_A-sentence φ, there exists some $p \in G$ such that either $p \Vdash \varphi$ or $p \Vdash \neg\varphi$.

We now deal with the existence of generic sets.

Lemma 6.2. Let $\mathcal{P} = \langle P, \leq, f\rangle$ be a forcing notion and $p \in P$. There is a generic set $G \subset P$ such that $p \in G$.

Proof. Let $(\phi_n)_{n\in\mathbb{N}}$ be an enumeration of all K_A sentences. We form recursively the sequence $(p_n)_{n\in\mathbb{N}}$ of conditions as follows: let $p_0 = p$; suppose that we have defined the finite sequence of conditions $p_0 \leq p_1 \leq \cdots \leq p_n$; if $p_n \Vdash \neg\phi_n$, then let $p_{n+1} = p_n$, and otherwise, choose some $p_{n+1} \geq p_n$ such that $p_{n+1} \Vdash \phi_n$. Let $G = \{q \in P : q \leq p_n$, for some $n \in \mathbb{N}\}$.

The set G is generic: if $q_0 \in G$ and $q_1 \leq q_0$, then there is some $n \in \mathbb{N}$ such that $q_0 \leq p_n$ and, therefore, $q_1 \leq p_n$, that is, $q_1 \in G$; if $q_0, q_1 \in G$, there are some $m, n \in \mathbb{N}$ such that $q_0 \leq p_m$ and $q_1 \leq p_n$, so $q_0, q_1 \leq r = p_{\max\{m,n\}}$; if ϕ_k is a K_A sentence and $p_k \nVdash \neg\phi$, then there is some $n \geq k$ such that $p_n \Vdash \phi_k$; if for no $q \in G$ $q \Vdash \phi_k$, then $p_k \Vdash \neg\phi$ (because otherwise there would be some $n \geq k$ such that $p_n \Vdash \phi_k$). □

We show in the following two lemmas that the generic sets produce models. The first gives maximal consistency and the second gives the properties used in Henkin's method of constants to build a model.

Lemma 6.3. Let $\mathcal{P} = \langle P, \leq, f\rangle$ be a forcing notion. The set T of all K_A-sentences forced by some $p \in G$ is a maximal consistent set of K_A-sentences.

Proof. We firstly tackle the consistency of T. If for some $p \in G$, $p \Vdash \phi$, then for all $q \in G$, there is some $r \in G$ with $p, q \leq r$ and so $r \Vdash \phi$. This implies that $q \nVdash \neg\phi$. Conversely, if for some $p \in G$, $p \Vdash \neg\phi$, there is no $q \in G$ such that $q \Vdash \phi$ because if $r \in G$ is such that $r \geq p, q$, then $r \nVdash \phi$ and, therefore, $q \nVdash \phi$. This implies that the set T is consistent.

The maximality of T comes from the third condition in the definition of generic set. □

Remark 6.4. The maximality of T implies that if $\phi, (\phi \to \psi) \in T$, then $\psi \in T$; and if $\phi(d) \in T$, then $\exists x\phi(x) \in T$; and if $\phi_k \in T$, then $\bigvee_{n\in\mathbb{N}} \phi_n \in T$.

Lemma 6.5. Let $\mathcal{P} = \langle P, \leq, f\rangle$ be a forcing notion, and T the set of all K_A-sentences forced by some $p \in G$. This set T has the following properties:

(1) $(\bigwedge_{n\in\mathbb{N}} \phi_n) \in T$ if, and only if, for all $n \in \mathbb{N}$ $\phi_n \in T$;
(2) $(\exists x\phi(x)) \in T$ if, and only if, for some new constant symbol $d \in D$ $\phi(d) \in T$;
(3) if $(\sigma = \tau) \in T$, for terms without variable σ and τ, then $(\tau = \sigma) \in T$;
(4) if $(\sigma = \tau), \phi(\sigma) \in T$, for terms without variable σ and τ, and sentence $\phi(\sigma)$, then $\phi(\tau) \in T$;

(5) for all terms τ there is some $d \in D$ such that $(d = \tau) \in T$.

Proof. Items (1), (2), (3) and (5) follow from the definition of forcing and the above remark. Item (4) follows from (2). □

Definition 6.6 (Generic Model)**.** Let T be an L-theory and D that set of new constant symbols. An $L(D)$-structure M is said to be T-*generic* if for all finite $p \subset \Delta(M)$, $T \cup p$ is consistent, and for each $L(D)$-sentence, $M \models \phi$ if, and only if, there is a finite $p \in \Delta(M)$ such that $p \Vdash \phi$.

We must assure that generic models do exist.

Theorem 6.7 (Generic Model)**.** Let $\langle P, \leq, f\rangle$ be a forcing notion and $p \in P$ a condition. There is a generic model for p.

Proof. Let $G \ni p$ be generic and let Σ be the set of all $L(D)$ sentences forced by some $q \in G$. We can do the Henkin construction with this Σ. □

We have a useful characterization of a generic model, from [1, Theorem 3.4, pp. 129-130].

Theorem 6.8. A model $M \models T^f$ is T-generic if, and only if, $T^f \cup \Delta(M)$ is complete.

Proof. □

Let T_0 be a consistent theory (set of sentences) in a countable admissible fragment of $L_{\omega_1\omega}$. Let P be the set of finite subsets of basic $L(C)$-sentences which are consistent with T_0, with the order $p \leq q$ if, and only if, $q \subseteq p$, and $f(p)$ is the subset of p containing all of its atomic sentences. The triple $(P, \leq, f)$ is a forcing notion.

The following result is taken from [15, Lemma 3, pp. 517-518]

Theorem 6.9 (Macintyre, [15])**.** The class of T^f-generic models can be axiomatised by an $L_{\omega_1\omega}$-sentence.

7. Robinson's Infinite Forcing

Let $\mathcal{M}$ be a non empty class of L-structure (for some signature L) and C a countable set of new constant symbols. A *basic formula* is an atomic $L(C)$-sentence or its negation.

Let $P(\mathcal{M})$ be the collection of finite sets, p, of basic formulas which are satisfiable in some $M \in \mathcal{M}$, for some interpretation of the new constants occurring in the formulas in p. Each such set $p \in P(\mathcal{M})$ is called a *condition*. We define the partial order $\leq$ in $P(\mathcal{M})$ by the reverse inclusion, $p \leq q$ if, and only if, $q \subseteq p$. The top element $\mathbf{1}$ is the empty set. We define the function $f : p \in P(\mathcal{M}) \mapsto f(p) = \{\phi \in p : \phi \text{is atomic}\}$.

8. EXERCISES

Exercise 8.1. Let $\mathcal{P} = \langle P, \leq, f\rangle$ be a forcing notion. Show that

(a) $p \Vdash \bigvee_{n\in\mathbb{N}} \phi_n$ if, and only if, for some $n \in \mathbb{N}$ $p \Vdash \phi_n$;

(b) $p \Vdash \forall x_i\phi$ if, and only if, for all $d \in D$, $p \Vdash \phi|_{x_i=d}$;

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