INROADS

Real Analysis Exchange
Vol. 49(2), 2024, pp. 1–11
DOI: 10.14321/realanalexch.49.2.1689708439

Oswaldo Rio Branco de Oliveira, Department of Mathematics, Universidade de São Paulo, São Paulo, SP 05508-090, Brasil.
email: oliveira@ime.usp.br

CHANGES OF VARIABLE FOR THE RIEMANN INTEGRAL ON THE REAL LINE

Abstract

We show two elementary versions of the *Change of Variable Theorem* (for the unidimensional Riemann integral) that are not a particular case of the influential general version due to Hyman Kestelman. Examples are given.

1 Introduction

The simplest version of the *Change of Variable for the Riemann Integral on the Real Line* theorem supposes that all the functions and derivatives involved in the formula are continuous on closed intervals.

Let us make a comment, based on Apostol [1, pp. 199–200]. The probably most general version of the *Change of Variable Theorem*, for the unidimensional Riemann integral, does not require the continuity of the main function involved and does not require that the derivative of the substitution map is continuous. Such well-known and influential general version has the form

$$\int_{G(\alpha)}^{G(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(G(t))\,g(t)\,dt,$$

with f integrable on the interval $G([\alpha,\beta])$ and the substitution map satisfying

$$G(t) = G(\alpha) + \int_{\alpha}^{t} g(\tau)\,d\tau, \quad \text{where } t \in [\alpha,\beta],$$

Mathematical Reviews subject classification: Primary: 26A42
Key words: Riemann integral, change of variable
Received by the editors July 18, 2023
Communicated by: Emma D'Aniello

for some function g integrable on $[\alpha, \beta]$. It is worth noting that Kestelman's version requires the integrability of functions f and g, the derivative of the substitution map G (the existence of the third and remaining integral follows, see [4]). It is also worth noting that the identity $G'(t) = g(t)$ is true on all the points of continuity of g but is not assured at any other point of $[\alpha, \beta]$.

The first proof of such general version is due to Kestelman [4], in 1961. In the same journal, same number, Davies [3] simplified Kestelman's proof. Since then, many articles have been published about this general version. See Bagby [2], Sarkhel and Výborný [7], and Torchinsky [8]. See also Torchinsky's book [9].

The versions proved in this article have the following features:

- The enunciates are very simple and the proofs are very elementary.
- They apply to some cases where Kestelman's general version does not.
- The first version is stronger than the textbooks' versions presented in, for instance, Knapp [5, pp. 37–38] (the proof of the first version has some similarities and many differences with respect to the proof in this book), and Rudin [6, p. 133].
- The integrability of the derivative of the substitution map is unnecessary.

2 Notation

Consider $[a, b] \subset \mathbb{R}$. We adopt partitions of the type

$$\mathcal{X} = \{a = x_0 \leqslant \cdots \leqslant x_n = b\},$$

thus allowing repeated points. We put $\Delta x_i = x_i - x_{i-1}$, for each $i = 1, \ldots, n$. The *norm* of $\mathcal{X}$ is defined by $|\mathcal{X}| = \max\{\Delta x_1, \ldots, \Delta x_n\}$.

Given a bounded function $f\colon [a, b] \to \mathbb{R}$ and a partition $\mathcal{X}$ of $[a, b]$, we write

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x),$$

for each $i = 1, \ldots, n$. We indicate the inferior and the superior *Riemann sums* of f with respect to the partition $\mathcal{X}$ by, respectively,

$$s(f, \mathcal{X}) = \sum_{i=1}^{n} m_i \Delta x_i, \quad S(f, \mathcal{X}) = \sum_{i=1}^{n} M_i \Delta x_i.$$

We notice that since f is bounded, the set of all inferior Riemann sums and the set of all superior Riemann sums are both bounded.

We say that the function f is *Riemann integrable* if we have the equality

$$\sup\{s(f,\mathcal{X})\} = \inf\{S(f,\mathcal{X})\},$$

where the infimum and the supremum are taken over all partitions $\mathcal{X}$ of $[a,b]$. In such case, the real number $\sup\{s(f,\mathcal{X})\} = \inf\{S(f,\mathcal{X})\}$ is called the *Riemann integral* of f and it is indicated by

$$\int_a^b f(x)\,dx.$$

Given an arbitrary real map $\varphi\colon [\alpha,\beta] \to \mathbb{R}$, we say that φ is *increasing* if we have $\varphi(t) \geqslant \varphi(\tau)$ for all $t \geqslant \tau$. Analogously, the map φ is *decreasing* if we have $\varphi(t) \leqslant \varphi(\tau)$ for all $t \geqslant \tau$. We say that φ is *monotone* if it is increasing, decreasing, or constant. Moreover, we say that φ is *piecewise monotone* if there exists a finite sequence $\{\alpha = t_0 < \cdots < t_N = \beta\}$ such that the map φ is monotone on each open sub-interval (t_j, t_{j+1}) for every $j = 0, \ldots, N-1$.

Given an arbitrary function $f\colon I \to \mathbb{R}$, with I an arbitrary real interval, we say that a function $F\colon I \to \mathbb{R}$ is a *primitive* of f if $F' = f$.

3 First theorem

Theorem 1. *Let $f\colon [a,b] \to \mathbb{R}$ be integrable and $\varphi\colon [\alpha,\beta] \to [a,b]$ be surjective, increasing (not necessarily strictly increasing) and continuous. Suppose that φ is differentiable on the open interval (α,β). The following are true:*

- *If φ' is integrable on $[\alpha,\beta]$, then the map $(f\circ\varphi)\,\varphi'$ is integrable on $[\alpha,\beta]$.*
- *If the product $(f\circ\varphi)\,\varphi'$ is integrable on $[\alpha,\beta]$, then*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt.$$

Proof. We split the proof into eight small and very easy steps.

1. *The hypothesis that φ is continuous is superfluous.* Let us show that the combined hypotheses φ surjective and φ increasing imply φ continuous. Since φ is surjective and increasing, we have $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. Thus, it is enough to show the continuity of the extension of φ given by

$$\Phi(t) = \begin{cases} a + (t-\alpha) & \text{if } t \leqslant \alpha, \\ \varphi(t) & \text{if } t \in [\alpha,\beta], \\ b + (t-\beta) & \text{if } t \geqslant \beta. \end{cases}$$

Clearly, $\Phi\colon \mathbb{R} \to \mathbb{R}$ is surjective and increasing. Let us pick an arbitrary point $t \in \mathbb{R}$ and $x = \Phi(t)$. Let x' and x'' be such that $x' < x < x''$. Since Φ is surjective, there exist t' and t'' such that $\Phi(t') = x'$ and $\Phi(t'') = x''$. Since Φ is increasing, we have $t' < t < t''$. Once more using that Φ is increasing, we conclude that $\Phi([t', t'']) \subset [x', x'']$. Thus, Φ is continuous.

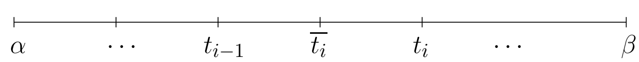
2. *The function φ is uniformly continuous.* Since φ is continuous on a closed and bounded interval, it follows that φ is uniformly continuous.

3. *We have $\varphi' \geqslant 0$ on the open interval (α, β).* It is true since φ is increasing.

4. The existence and also the value of the Riemann integral of a function do not change if we redefine such function at a finite set of points. Thus, in order to investigate the Riemann integrals of φ' and of the product $(f \circ \varphi)\,\varphi'$ on the interval $[\alpha, \beta]$, we may define $\varphi'(\alpha)$ and $\varphi'(\beta)$ at will.

5. Let $\mathcal{T} = \{\alpha = t_0 \leqslant \cdots \leqslant t_n = \beta\}$ be an arbitrary partition of the interval $[\alpha, \beta]$ and $\mathcal{X} = \{a = x_0 \leqslant \cdots \leqslant x_n = b\}$ be the partition of the interval $[a, b]$ given by $\mathcal{X} = \varphi(\mathcal{T})$. That is, suppose $x_i = \varphi(t_i)$ for each $i = 0, \ldots, n$. The mean-value theorem yields a point $\overline{t_i} \in [t_{i-1}, t_i]$ (we remark that if $t_{i-1} < t_i$, then $\overline{t_i} \in (t_{i-1}, t_i)$) satisfying

$$\Delta x_i = \varphi(t_i) - \varphi(t_{i-1}) = \varphi'(\overline{t_i})\,\Delta t_i.$$



6. *If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$.* It follows from the uniform continuity of φ.

7. *If φ' is integrable, then $(f \circ \varphi)\,\varphi'$ is integrable.* Let τ_i be arbitrary in $[t_{i-1}, t_i]$. We notice that $\varphi(\tau_i) \in [x_{i-1}, x_i]$. In what follows, for simplicity, we omit the summation index. Let us look at the Riemann sum

$$\sum f(\varphi(\tau_i))\,\varphi'(\tau_i)\,\Delta t_i = \sum f(\varphi(\tau_i))\,\Delta x_i + \sum f(\varphi(\tau_i))\,\big[\varphi'(\tau_i)) - \varphi'(\overline{t_i})\big]\,\Delta t_i.$$

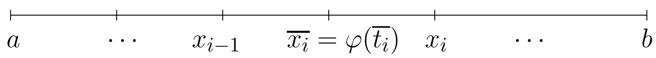
If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the first sum on the right goes to $\int_a^b f\,dx$.

Let M be a constant such that $|f| \leqslant M$ (obviously, f is bounded). Then,

$$\Big|\sum f(\varphi(\tau_i))\big[\varphi'(\tau_i)) - \varphi'(\overline{t_i})\big]\,\Delta t_i\Big| \leqslant M\big[S(\varphi', \mathcal{T}) - s(\varphi', \mathcal{T})\big] \xrightarrow{|\mathcal{T}|\to 0} 0.$$

Therefore, the function $(f \circ \varphi)\,\varphi'$ is integrable (the value of its integral equals the one of f).

8. *If* $(f \circ \varphi)\,\varphi'$ *is integrable, then the value of its integral equals the one of* f. With the above notation, we choose $\tau_i = \overline{t_i}$ and write $\overline{x_i} = \varphi(\overline{t_i})$.



Hence, we have

$$\sum f(\varphi(\overline{t_i}))\,\varphi'(\overline{t_i})\,\Delta t_i = \sum f(\overline{x_i})\,\Delta x_i.$$

If $|\mathcal{T}| \to 0$, by definition the left side goes to the integral of $(f \circ \varphi)\,\varphi'$.
If $|\mathcal{T}| \to 0$, then $|\mathcal{X}| \to 0$ and the right side goes to the integral of f. □

Corollary 2. *Keeping all the other hypotheses of Theorem 1, let us suppose that the surjective and continuous map* $\varphi\colon [\alpha, \beta] \to [a, b]$ *satisfies one of the following five conditions:*

(a) φ *is monotone.*

(b) φ *is piecewise monotone.*

(c) φ *is piecewise monotone on* $[\alpha + \epsilon, \beta]$, *for each* $0 < \epsilon < \beta - \alpha$.

(d) φ' *has a finite number of zeros.*

(e) φ' *has a finite number of zeros in* $[\alpha + \epsilon, \beta)$, *for each* $0 < \epsilon < \beta - \alpha$.

Then, the following two claims are true:

- *If* φ' *is integrable, then* $(f \circ \varphi)\,\varphi'$ *also is.*
- *If* $(f \circ \varphi)\,\varphi'$ *is integrable, then*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt.$$

PROOF. We split the proof with respect to the five conditions.
(a) The case φ is increasing is already proven. Suppose φ is decreasing. Then

$$\psi(s) = \varphi(\alpha + \beta - s), \quad \text{where } s \in [\alpha, \beta],$$

is increasing, with $\psi(\alpha) = \varphi(\beta) = a$ and $\psi(\beta) = \varphi(\alpha) = b$.

It is clear that the change of variable φ is integrable if and only if the change of variable ψ is integrable. Analogously for the pair of derivatives φ' and ψ', and for the pair of functions $(f\circ\varphi)\,\varphi'$ and $(f\circ\psi)\,\psi'$.

Applying the formula in Theorem 1 to the function $(f\circ\psi)\,\psi'$, we find

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\psi(s))\,\psi'(s)\,ds = -\int_\alpha^\beta f\big(\varphi(\alpha+\beta-s)\big)\,\varphi'(\alpha+\beta-s)\,ds.$$

Returning to the variable $t=\alpha+\beta-s$, we conclude this case with

$$\int_a^b f(x)\,dx = -\int_\beta^\alpha f(\varphi(t))[-\varphi'(t)]\,dt = \int_\beta^\alpha f(\varphi(t))\,\varphi'(t)\,dt.$$

(b) In this case, there exists a finite sequence $\{\alpha_0=\alpha<\alpha_1<\cdots<\alpha_N=\beta\}$ such that φ is monotone on each closed sub-interval $[\alpha_j,\alpha_{j+1}]$, where $j=0,\ldots,N-1$. Hence, for what we have already seen in (a), we have

$$\sum_{j=0}^{N-1}\int_{\varphi(\alpha_j)}^{\varphi(\alpha_{j+1})} f(x)\,dx = \sum_{j=0}^{N-1}\int_{\alpha_j}^{\alpha_{j+1}} f(\varphi(t))\,\varphi'(t)\,dt.$$

From which it follows, and thus completing this case, that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt.$$

(c) It follows from case (b), applied to the sub-interval $[\alpha+\epsilon,\beta]$, from the continuity of the integral with respect to the end points of integration, and the continuity of φ. Writing down, we have

$$\int_{\varphi(\alpha+\epsilon)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha+\epsilon}^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

and thus letting $\epsilon\to 0$ we obtain the desired formula.

(d) Let $\{\alpha_1<\cdots<\alpha_{N-1}\}$, where $N\geqslant 2$, be the set of zeros of the derivative φ', if there are any. Let us put $\alpha_0=\alpha$ and $\alpha_N=\beta$. Fixing an arbitrary sub-interval (α_j,α_{j+1}), with $j\in\{0,\ldots,N-1\}$, by the *Intermediate Value Theorem for Derivatives* (*Darboux's Theorem*) the derivative φ' assumes only one sign (either strictly positive or strictly negative) along it. Hence, along this open sub-interval the change of variable φ is either strictly increasing or strictly decreasing. This shows that the map φ is piecewise monotone. Then, the conclusion is immediate from (b).

(e) From case (d) it follows that

$$\int_{\varphi(\alpha+\epsilon)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha+\epsilon}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt, \quad \text{for each } 0<\epsilon<\beta-\alpha.$$

Let us split the analysis of this integral identity into two sub-cases.

The sub-case $(f\circ\varphi)\,\varphi'$ *integrable* (*on* $[\alpha,\beta]$). From the continuity of an integral with regard to the endpoints and the continuity of φ, we obtain

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt.$$

This sub-case is done. Let us return to the integral identity in question.

The sub-case φ' *integrable* (*on* $[\alpha,\beta]$). Here we use the concept of *zero measure*. The well-known *Lebesgue's Criterion for Riemann Integrability* assures us that the set of discontinuities of $(f\circ\varphi)\,\varphi'$ in the sub-interval $[\alpha+\epsilon,\beta]$ has measure zero, for each allowed ϵ. Hence, it is not difficult to see that the set of discontinuities of the function $(f\circ\varphi)\,\varphi'$ in the interval $[\alpha,\beta]$ also has measure zero. On the other hand, since f and φ' are both integrable on their respective domains, we see that f and φ' are both bounded. Thus, by *Lebesgue's Criterion for Riemann Integrability*, the function $(f\circ\varphi)\,\varphi'$ is integrable. This sub-case is done. □

4 Two examples

Example 3. (An example for Theorem 1) *Let us consider the following pair of functions:*

$$f(x)=x \;\text{ for all }\; x\in[0,1], \quad \text{and} \quad \varphi(t)=\sqrt{t} \;\text{ for all }\; t\in[0,1].$$

Evidently, the function f is integrable. Moreover, the map $\varphi\colon[0,1]\to[0,1]$ is surjective, increasing, and continuous. Yet, the derivative φ' is defined on the half-open interval $(0,1]$ and

$$\varphi'(t)=\frac{1}{2\sqrt{t}}.$$

It is clear that φ' is not bounded on $(0,1]$ and thus *not integrable* on $[0,1]$ (as already commented in Theorem 1, step 4, we may define $\varphi'(0)$ at will).

It is also clear that the function

$$f(\varphi(t))\,\varphi'(t)=\frac{\sqrt{t}}{2\sqrt{t}}=\frac{1}{2}, \quad \text{for } t\neq 0,$$

is integrable on $[0,1]$. Hence, from Theorem 1 it follows that

$$\int_0^1 x\,dx = \int_0^1 \frac{1}{2}\,dt.$$

On other hand, φ' is not integrable on $[0,1]$ and we cannot write

$$\sqrt{t} = \int_0^t \frac{1}{2\sqrt{\tau}}\,d\tau, \quad \text{for all } \ t \in [0,1].$$

Therefore, Kestelman's version (with $g = \varphi'$) does not apply in this case.

Example 4. (An example for Corollary 2) *Let us consider the pair*

$$f(x) = x^3, \ \ \textit{if} \ x \in \left[0, \frac{2}{\pi}\right], \quad \textit{and} \quad \varphi(t) = \begin{cases} 0, & \textit{if} \ t = 0, \\ t \sin \frac{1}{t}, & \textit{if} \ t \in \left(0, \frac{2}{\pi}\right]. \end{cases}$$

The function f is evidently integrable. The map φ is continuous and, as is well known, oscillates near the origin.

Let us pick ϵ, with $0 < \epsilon < 2/\pi$. Then, in the sub-interval $[0,\epsilon]$, the map φ has an infinite number of points of local maximum, as well as an infinite number of points of local minimum. It then follows that the derivative φ' has an infinite number of zeros in the interval $[0,\epsilon]$ and, it is not difficult to see, the derivative φ' has a finite number of zeros in the interval $[\epsilon, 2/\pi]$.

The derivative φ' is defined on the half-open interval $(0, 2/\pi]$ and we have

$$\varphi'(t) = \sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}.$$

Obviously, φ' is unbounded and thus *not integrable* on $[0, 2/\pi]$.

On other hand, it is not hard to show the integrability of the function

$$(f \circ \varphi)(t)\,\varphi'(t) = t^3\left(\sin^3\frac{1}{t}\right)\left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right).$$

By Corollary 2, item (d), we find

$$\int_0^{2/\pi} x^3\,dx = \int_0^{2/\pi} [\varphi(t)]^3\,\varphi'(t)\,dt.$$

From which follows

$$\int_0^{2/\pi} x^3\,dx = \frac{\varphi^4(t)}{4}\Big|_0^{2/\pi} = \frac{4}{\pi^4}.$$

Since the derivative φ' is not integrable on $[0, 2/\pi]$, Kestelman's general version does not apply to this example.

5 Second theorem

We emphasize that the theorem that follows does not require either the monotocity of the substitution map φ or the integrability of its derivative φ'.

Theorem 5. *Let us consider an integrable function $f\colon I \to \mathbb{R}$, with I an interval, and a map $\varphi\colon [\alpha,\beta] \to I$. Let us suppose that f has a primitive. Let us also suppose that φ is continuous on $[\alpha,\beta]$ and differentiable on (α,β). The following is true:*

- *If the product $(f\circ\varphi)\,\varphi'$ is integrable on $[\alpha,\beta]$, then we have the formula*

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt.$$

Proof. If the lateral derivatives $\varphi'(\alpha)$ and $\varphi'(\beta)$ do not exist then we may define them at will, as already seen in the proof of Theorem 1. This does not affect the existence or the value of the integral of the product $(f\circ\varphi)\,\varphi'$.

Now, let F be a primitive of the function f and let $\epsilon > 0$ be small enough. Then we have

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} f(\varphi(t))\,\varphi'(t)\,dt = \int_{\alpha+\epsilon}^{\beta-\epsilon} F'(\varphi(t))\,\varphi'(t)\,dt = \int_{\alpha+\epsilon}^{\beta-\epsilon} (F\circ\varphi)'(t)\,dt$$
$$= F(\varphi(\beta-\epsilon)) - F(\varphi(\alpha+\epsilon)) = \int_{\varphi(\alpha+\epsilon)}^{\varphi(\beta-\epsilon)} f(x)\,dx.$$

Therefore, letting $\epsilon \to 0$, by the continuity of φ and by the continuity of the integral with respect to the integration endpoints we conclude that

$$\int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx.$$

□

We notice the following:

- If $f\colon I \to \mathbb{R}$ is a continuous function, with I an interval on the real line, it is well known that f has a primitive.
- The two examples in Section 4 are suitable for Theorem 5.

6 Some final remarks

The author humbly hopes that these proofs may be quite useful in classrooms and for further research on the Change of Variable Formula.

Acknowledgments. The author is very grateful to the referees for their critique and to Alberto Torchinsky, Tepper L. Gill, Paul Humke, Alexandre Lymberopoulos, and Antônio L. Pereira for their very kind comments and encouragement along the way.

References

[1] T. M. Apostol, *Análisis Matemático*, Second edition, Editorial Reverté, Barcelona, 1977.

[2] R. J. Bagby, *The substitution theorem for Riemann integrals*, Real Anal. Exchange, **27(1)** (2001/2002), 309–314.

[3] R. O. Davies, *An elementary proof of the theorem on change of variable in Riemann integration*, Math. Gaz., **45(351)** (1961), 23–25.

[4] H. Kestelman, *Change of variable in Riemann integration*, Math. Gaz., **45(351)** (1961), 17–23.

[5] A. W. Knapp, *Basic Real Analysis*, Birkhäuser, Boston, 2005.

[6] W. Rudin, *Principles of Mathematical Analysis*, Third edition. McGraw-Hill Book, Singapore, 1976.

[7] D. N. Sarkhel and R. Výborný, *A change of variables theorem for the Riemann integral*, Real Anal. Exchange, **22(1)** (1996/97), 390–395.

[8] A. Torchinsky, *Change of variable formulas for Riemann integrals*, Real Anal. Exchange, **45(1)** (2020), 151–172.

[9] A. Torchinsky, *A Modern View of the Riemann Integral*. Lecture Notes in Mathematics, **2309**, Springer, Cham, 2022.