A ZERO POLYNOMIAL IS A NULL POLYNOMIAL

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This text shows, very elementarily and without employing differentiability arguments or continuity arguments, that a zero real polynomial is a null polynomial.

Let $P = P(x)$ be a polynomial with real coefficients and x a real variable.

We say that P is a *zero polynomial* if we have

$$P(x) = 0, \text{ for all } x \in \mathbb{R}.$$

We say that P is a *null polynomial* if its coefficients are all zero.

If we employ differentiability, it is very trivial to show that a zero polynomial is a null polynomial. One just has to differentiate the equation $P(x) = 0$ a right amount of times. However, the concept "differentiability" is quite sophisticated for the task and thus it is natural to look for a more basic proof.

If we employ continuity, we can also produce a trivial proof. Not so easy as the one using differentiability, but still easy. Such proof is very worth to know.

The main proof in this note avoids limits, continuity and differentiability.

Right after such main proof, we present a proof that employs differentiability and a proof that employs continuity.

Let us begin.

Notation. Let us fix a natural number $n \in \{0, 1, 2, \ldots\}$ and $n+1$ real coefficients $a_0, \ldots, a_n$. Let us consider a real polynomial

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \text{ where } x \in \mathbb{R}.$$

Theorem. *Let us suppose that the real polynomial $P(x)$ satisfies*

$$P(x) = 0 \textit{ for all } x \in \mathbb{R}.$$

Then, all the coefficients of the polynomial P are zero.

Proof. The cases $n = 0$, $n = 1$, $n = 2$ and $n = 3$ are illuminating.

- The case $n = 0$. This is very obvious.

- The case $n = 1$. Then we have

  $$a_0 + a_1 x = 0, \text{ for all } x \in (-\infty, +\infty).$$

  Substituting $x = 0$, we obtain

  $$\boxed{a_0 = 0}.$$

  Hence we have

  $$P(x) = a_1 x = 0 \text{ for all } x \in (-\infty, +\infty).$$

  Substituting $x = 1$ we find

  $$\boxed{a_1 = 0}.$$

  This case is complete.

- The case $n = 2$. Then we have

  $$a_0 + a_1 x + a_2 x^2 = 0, \text{ for all} x \in (-\infty, +\infty).$$

  Substituting $x = 0$, we obtain

  $$\boxed{a_0 = 0}.$$

Hence we have

$$P(x) = a_1 x + a_2 x^2 = 0 \text{ for all } x \in (-\infty, +\infty).$$

Substituting $x = 1$ and $x = -1$, we obtain

$$\begin{cases} a_1 + a_2 = 0 \\ -a_1 + a_2 = 0. \end{cases}$$

This system is trivial and we arrive at

$$\boxed{a_1 = a_2 = 0}.$$

The case $n = 2$ is complete.

- The case $n = 3$. This is the main case.

  Let us simplify things and write the polynomial as

$$P(x) = a + bx + cx^2 + dx^3,$$

where a, b, c and d are real numbers.

Since we have $P(x) = 0$ for all $x \in \mathbb{R}$, substituting $x = 0$ we obtain

$$\boxed{a = 0}.$$

Thus, we have

$$P(x) = bx + cx^2 + dx^3 = 0, \text{ for all } x \in (-\infty, +\infty).$$

From which follows

$$x(b + cx + dx^2) = 0, \text{ for all } x \in (-\infty, +\infty).$$

Now, dividing by $x \neq 0$ we find

$$b + cx + dx^2 = 0, \text{ for all } x \neq 0.$$

Let us prove that $b = 0$.

Let us pick ϵ with $0 < \epsilon < 1$.

Next, we consider a nonzero x such that

$$0 < |x| < \frac{\epsilon/2}{(1+|c|)(1+|d|)}.$$

It is not hard to see that such x satisfies

$$|cx| \le \frac{\epsilon}{2}.$$

Similarly, noticing that $|x| \le 1$, we also have

$$|dx^2| \le |dx| \le \frac{\epsilon}{2}.$$

Thus, we arrive at the two inequalities

$$b - \frac{\epsilon}{2} - \frac{\epsilon}{2} \le \; b + cx + dx^2 \; \le b + \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

By hypothesis, we have (notice that $x \neq 0$)

$$b + cx + dx^2 = 0.$$

Substituting this identity into the last two inequalities we obtain

$$b - \epsilon \le 0 \; \le b + \epsilon.$$

That is,

$$-\epsilon \le -b \le \epsilon$$

and

$$|b| \le \epsilon.$$

Hence, we have established the inequality $|b| \le \epsilon$ for every $\epsilon > 0$.

Therefore, by the *Trichotomy Property* we may conclude that

$$\boxed{b = 0}.$$

This fact ($b = 0$) is the main point of such case (the case $n = 3$).

Let us finish the case $n = 3$. At this stage we have $a = b = 0$ and

$$cx + dx^2 = 0 \text{ for all } x \in (-\infty, +\infty).$$

This implies (dividing by $x \neq 0$)

$$c + dx = 0 \text{ for all } x \neq 0.$$

Now, as in the case $n = 2$, we obtain (by substituting $x = 1$ and $x = -1$)

$$\boxed{c = d = 0}.$$

The case $n = 3$ is complete.

- The general case. It can be proven analogously to the case $n = 3$. I leave it to the reader♣

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