Woodall's conjecture and series-parallel digraphs

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Abstract

Woodall conjectured that the size of a smallest dicut of a digraph is equal to the size of a largest set of pairwise disjoint transversals of dicuts. Lee and Wakabayashi proved that the conjecture is true when restricted to series-parallel digraphs. Their proof is indirect, as it deals with the dual of the conjecture, which concerns dicircuits instead of dicuts. The present paper gives a (rather verbose) direct proof.

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1 Introduction

The paper is organized as follows. Section 2 proves two properties of series-parallel graphs. Section 3 defines dicuts and dijoins of digraphs. Section 4 states Woodall's conjecture relating minimum dicuts to maximum sets of pairwise disjoint dijoins. Section 5 adds capacities to the arcs of the digraphs and states the corresponding generalization of Woodall's conjecture. The section also introduces the idea of critical capacities and shows some simple properties of such capacities. Section 6 states the Lee–Wakabayashi theorem, which verifies the restriction of Woodall's conjecture to series-parallel digraphs. The section also sketches a reduction of the theorem to 2-connected acyclic digraphs. Section 7 gives a proof of the Lee–Wakabayashi theorem. An appendix summarizes graph theory terminology and establishes some notational conventions. Finally, an index points to the definitions of all the technical terms.

2 Series-parallel graphs

A ***graph*** is a pair (V, E) of sets where V is a finite set of ***vertices*** and E is some set of unordered pairs of vertices. Each element of E is an ***edge***. According to this definition, a graph has no loops and no parallel edges.

To ***subdivide*** an edge of a graph H is to replace that edge by two edges "in series". More precisely, to subdivide an edge yz is to add a new vertex r to H and to replace yz with the new edges yr and rz. A ***subdivision*** of a graph H is any graph obtained by recursively subdividing edges of H.

A graph G is ***K4-free***, or ***series-parallel***, if it does not contain a subdivision of a complete graph on 4 vertices, i.e., if no subgraph of G is a subdivision of a K_4.

EXAMPLES: Any graph that consists of a circuit and nothing else is K4-free. For any graph G consisting of a circuit of length greater than 3 and any two nonadjacent vertices v and w of G, the graph $G + vw$ is K4-free.

Lemma 2.1 (Duffin [Duf65]) *Every 2-connected K4-free graph has a vertex of degree* 2.

PROOF: Let G be a 2-connected graph. (Of course G has at least 3 vertices.) Let $d(v)$ denote the degree of a vertex v of G. Suppose that there is no vertex v such that $d(v) = 2$; we will show that G is not K4-free.

Of course $d(v) \geq 2$ for every vertex v. Hence G has a circuit. Let C be a circuit of maximum length. Let's say that a path ***across*** C is any path of nonnull length in $G - E(C)$ that has origin and terminus in $V(C)$ but no internal vertices in $V(C)$. We claim that every vertex of C is the origin of a path across C. Here is a proof of this claim:

Let r be a vertex of C and let R be the set of termini of all the paths in $G - E(C)$ that have origin r and no vertices in C other than the origin. Let ∂R be the set of all edges of G that have exactly one end in R. Suppose for a moment that $\partial R \setminus E(C) = \emptyset$. Then either $R = \{r\}$, in which case $d(r) = 2$, or $R \neq \{r\}$, in which case $G - r$ is disconnected, i.e., r is a cut vertex. Both alternatives are contrary to our hypotheses. Hence, we must have $\partial R \setminus E(C) \neq \emptyset$. Since $r \in R$ and every edge in $\partial R \setminus E(C)$ has one end in R and the other in C, there exist a path across C with origin r, as claimed.

Let P be any path across C and let r be the origin and s the terminus of P. Let $C_0(P)$ be the segment of C running from r to s and let $C_1(P)$ be the complement of C_0 in C i.e., the segment of C running from s to r. Of course $C_0(P)$ and $C_1(P)$ are paths of G. Let $l(L)$ denote the length of any path L and adjust notation so that the $l(C_0(P)) \leq l(C_1(P))$. Choose P so that $l(C_0(P))$ is as small as possible. If $l(C_0(P)) = 1$ then $l(P) \geq 2$ and therefore the circuit that results from concatenating P and $C_1(P)$ will be longer than C, contrary to our choice of C. We conclude that $l(C_0(P)) \geq 2$ and therefore $C_0(P)$ has an internal vertex, say x. As shown in the previous paragraph, x is the origin of a path Q across C. Let z be the terminus of Q. Suppose for a moment that z is a vertex of $C_0(P)$. Then $C_0(Q)$ is a segment of $C_0(P)$ and $l(C_0(Q)) < l(C_0(P))$, contrary to our choice of P. This contradiction shows that z must be an internal vertex of $C_1(P)$. But then the circuit C together with the paths P and Q constitute a subdivision of a K_4 in G. The existence of such a subgraph proves that G is not K4-free. □

A ***triplet*** in a graph G is a vertex of degree 2 together with its two neighbors. More precisely, a triplet is a sequence (y, r, z) of three vertices of G such that yr and rz are edges of G. We say that r is the ***head*** of the triplet while y and z are the ***feet***. The edges yr and rz are the ***legs*** of the triplet. A triplet (y, r, z) is ***closed*** if yz is an edge of G and ***open*** otherwise. If the triplet is closed, the edge yz is the ***base*** of the triplet.

Two triplets, (y, r, z) and (p, s, q), ***overlap*** if $(r, z) = (p, s)$. If triplets (y, r, z) and (p, s, q) overlap then, clearly, both are open.

An ***ear*** of a graph is a path of nonnull length whose internal vertices have degree 2. Every triplet is, of course, an ear. Moreover, if v_i is an interval vertex of ear $(v_0, v_1, v_2, \ldots, v_k)$ then (v_{i-1}, v_i, v_{i+1}) is a triplet. Hence, an ear is a sequence of overlapping triplets, the number of triplets being equal to the number of internal vertices of the ear.

DRAFT

Of particular interest are pairs of ears with a common origin and a common terminus. The two ears of any such pair are, of course, internally disjoint.

Lemma 2.2 (pair of ears) *Every 2-connected K4-free graph has two ears with a common origin and a common terminus.*

PROOF: Let G be a 2-connected K4-free graph. According to lemma 2.1, G has a vertex of degree 2 and therefore a triplet, say (y, r, z). If the triplet is closed — this is certainly the case if G has only 3 vertices — then (y, r, z) and (y, z) are the required ears. Next, suppose that the triplet is open, i.e., that yz is not an edge of G. Let G' be the graph $(G - r) + yz$. Clearly G' is K4-free. Moreover, G' is 2-connected, as we proceed to show.

Let s and t be any two vertices of G'. Since G is 2-connected, there are two internally disjoint paths, say S and T, from s to t in G. If r is neither a vertex of S nor a vertex of T then S and T are internally disjoint in G'. Suppose next that r is a vertex of S or T, say a vertex of S. Since r is not in G', it is internal to S, and therefore (y, r, z) is a segment of S. Replace the segment (y, r, z) with (y, z) to obtain a path S'. Then S' and T are internally disjoint in G'. Moreover, $S' \neq T$ since otherwise S and T would be (y, r, z) and (y, z) respectively and yz would be an edge of G, contrary to our hypothesis. Hence, G' is 2-connected, as claimed.

Since G' is 2-connected and K4-free, and since $V(G') \subset V(G)$, we may assume, as induction hypothesis, that G' has two ears, say P' and Q', with common origin and common terminus. If yz is not an edge of P' or Q' then P' and Q' are ears in G, as required. Next suppose that yz is an edge of P' or Q', say an edge of P'. Replace the segment (y, z) of P' with the triplet (y, r, z) to obtain a path P. Now P and Q' are the required ears in G. □

3 Digraphs, cuts, and joins

A ***digraph*** is a pair (V, A) of sets where V is a finite set of ***vertices*** and A is some set of ordered pairs of distinct vertices. Each element of A is an ***arc***. Any arc of the form (v, w) will be denoted by vw. The ***positive end*** of vw is v and the ***negative end*** is w. The sets of vertices and arcs of a digraph D will be denoted by $V(D)$ and $A(D)$ respectively. The digraph D is ***empty*** if $A(D) = \emptyset$.

We add an unusual requirement to the definition: our digraphs have no *antiparallel* arcs. In other words, for any two vertices v and w, at most one of the pairs (v, w) and (w, v) is an arc.

The ***underlying graph*** of a digraph is obtained by replacing each arc vw of D with the edge vw. In other words, a graph G underlies a digraph D if $V(G) = V(D)$ and G has an edge vw for each arc vw of D. Of course $|E(G)| = |A(D)|$.

An ***orientation*** of a graph G is any digraph D obtained by choosing one of the two possible orientations for each edge of G. In other words, an orientation of G is any digraph D obtained by replacing each edge vw of G with the arc vw or the arc wv. A digraph D is an orientation of a graph G if and only if G is the graph underlying D.

Two distinct vertices v and w of a digraph are ***adjacent*** if they are adjacent in the underlying graph. The ***degree*** of a vertex in the digraph is the degree of the vertex in the underlying graph. A digraph is ***connected*** if its underlying graph is connected.

DRAFT

A ***path*** in a digraph is a path in its underlying graph. Similarly, a ***circuit*** in a digraph is a circuit in its underlying graph. For any path or a circuit $(v_0, v_1, \ldots, v_k)$ in a digraph, any arc of the form $v_{i-1}v_i$ is ***forward-directed*** and any arc of the form $v_i v_{i-1}$ is ***backward-directed***.

A path in a digraph is ***directed*** if all its arcs are forward-directed. Similarly, a circuit is ***directed*** if all its arcs are forward-directed. A digraph is ***acyclic*** if it has no directed circuits. Acyclic digraphs are also known as ***dags***. For any two distinct vertices s and t of an acyclic digraph, if there is a directed path from s to t then there is no directed path from t to s.

The ***transpose***, or ***directional dual***, of a digraph D is the digraph $\tilde{D}$ obtained by replacing each arc vw of D with the ordered pair (w, v). Of course D is acyclic if and only if $\tilde{D}$ is acyclic.

For any arc a of a digraph D, we denote by $D-a$ the digraph $(V(D), A(D)-a)$. For any two nonadjacent vertices v and w of D, we denote by $D + vw$ the digraph $(V(D), A(D) + vw)$. For any vertex v of a digraph D, we denote by $D - v$ the digraph whose set of vertices is $V(D) - v$ and whose arcs are all the arcs of D that have no end equal to v.

3.1 Cuts

For any set X of vertices of a digraph D, we denote by ∂X the set of all the arcs that have exactly one end in X. Of course $\partial X = \partial(V(D) \setminus X)$. The set X is ***trivial*** if $\partial X = \emptyset$. The sets $\emptyset$ and $V(D)$ are, of course, trivial. These are the only trivial sets if and only if D is connected.

An arc vw ***leaves*** a set X if $v \in X$ and $w \in V(D) \setminus X$. An arc vw ***enters*** X if it leaves $V(D) \setminus X$. A ***source*** is a vertex r such that no arc enters $\{r\}$ and a ***sink*** is a vertex s such that no arc leaves $\{s\}$. More generally, a ***source*** is a set of vertices that no arc enters and a ***sink*** is a set of vertices that no arc leaves.

A ***dicut***, or simply ***cut***, of a digraph is any nonempty set of the form ∂S where S is a source or a sink. In other words, a cut is a set ∂S such that S is a nontrivial source or a nontrivial sink. We say that the cut ∂S is ***associated with*** S. The ***positive shore*** of a cut C is any source S such that $\partial S = C$. The ***negative shore*** of C is any sink S such that $\partial S = C$. In a connected digraph, every cut has a unique positive shore and a unique negative shore.

A cut ∂S ***separates*** a vertex x ***from*** a vertex y if $x \in S$ and $y \notin S$. More generally, a cut ∂S ***separates*** a set X of vertices from a set Y if $X \subseteq S$ and $Y \cap S = \emptyset$.

EXAMPLES: Let D be a digraph consisting of a path of nonnull length. Then, for every arc a of D, the set $\{a\}$ is a cut. Now let D be a digraph consisting of a directed circuit. Then D has no cuts.

Lemma 3.1 (dichotomy) *Every arc of a digraph belongs to a cut or to a directed circuit, but not both.*

PROOF: Let vw be an arc of a digraph. Let S be the set of termini of all directed paths with origin w. Of course S is a sink. If $v \in S$ then vw belongs to a directed circuit. Otherwise, $vw \in \partial S$ and ∂S is a cut. □

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3.2 Joins

A ***dijoin***, or simply ***join***, of a digraph D is any set of arcs that intersects every cut. In other words, a join of D is any subset J of $A(D)$ such that $J \cap C \neq \emptyset$ for every cut C. In any digraph D, the set $A(D)$ is a join. A join J is ***minimal*** if no proper subset of J is a join.

Lemma 3.2 *In any connected digraph, a set J of arcs is a join if and only if, for every vertex s and every vertex t, there is a path from s to t whose forward-directed arcs belong to J.*

PROOF: Suppose J is a join of a connected digraph and let s and t be two vertices of the digraph. Let S be the set of termini of all paths with origin s whose forward-directed arcs are in J. All we need to show is that $t \in S$. No arc enters S and therefore S is a source. Moreover, no arc of J leaves S, whence $\partial S \cap J = \emptyset$. Since J is a join, $\partial S = \emptyset$. Since the digraph is connected, $S = V(G)$ and so $t \in S$.

Now suppose that J is any set of arcs such that, for every vertex s and every vertex t, there is a path from s to t whose forward-directed arcs belong to J. All we need to show is that $J \cap C \neq \emptyset$ for every cut C. Let S the positive shore of a cut. Since $\partial S \neq \emptyset$, there is vertex s in S and a vertex t in $V(D) \setminus S$. Let P be a path from s to t whose forward-directed arcs belong to J. Of course some arc a of P belongs to ∂S. Since S is a source, a is forward-directed in P and therefore $a \in J$. Hence $J \cap \partial S \neq \emptyset$. □

4 Woodall's conjecture

A ***packing*** of joins of a digraph is a disjoint set of joins, i.e., a set of joins such that each arc of the digraph belongs to at most one of the joins of the set. The following lemma establishes an obvious inequality:

Lemma 4.1 *In any digraph, for any cut C and any packing $\mathcal{P}$ of joins, the inequality $|\mathcal{P}| \leq |C|$ holds.* □

Woodall conjectured [Woo78a, Woo78b] that

Conjecture 1 (Woodall) *Every digraph has a packing $\mathcal{P}$ of joins and a cut C such that $|\mathcal{P}| = |C|$.*

A cut C of a digraph is ***minimum*** if there is no cut C' such that $|C'| < |C|$. A packing $\mathcal{P}$ of joins is ***maximum*** if there is no packing $\mathcal{P}'$ such that $|\mathcal{P}'| > |\mathcal{P}|$. The size of a minimum cut and the size of a maximum packing of joins of a digraph D are denoted by

$$\tau(D) \text{ and } \nu(D)$$

respectively. If D has no cuts then $\tau(D) = \infty$ and $\emptyset$ is a join, whence $\nu(D) = \infty$. Otherwise, both $\tau(D)$ and $\nu(D)$ are finite and $\tau(D) > 0$.

Lemma 4.1 is equivalent to the following statement: $\nu(D) \leq \tau(D)$ for every digraph D. Conjecture 1 can be stated as follows: $\nu(D) = \tau(D)$ for every digraph D.

DRAFT

EXAMPLE: Let D be an acyclic orientation of a graph that consists of a circuit and nothing else. Then $\tau(D) = 2$. If J_1 is the set of forward-directed arcs of the circuit and J_2 is the set of backward-directed arcs then $\{J_1, J_2\}$ is a packing of joins of D. Hence, $\nu(D) = 2$.

By virtue of lemma 3.1, conjecture 1 holds if $\nu(D) = \tau(D)$ for every dag D. Hence, it is enough to prove the conjecture for dags.

5 Conjecture for capacitated digraphs

A ***capacity vector***, or ***upper bound vector***, for a digraph D is any vector u indexed by $A(D)$ with values in the set $\{0, 1, 2, 3, \ldots\}$ of natural numbers. For any arc a, the number u_a is the ***capacity*** of a. The ***capacity*** of a set C of arcs is the number $u(C) := \sum_{a \in C} u_a$.

A ***capacitated digraph*** is a pair (D, u) where D is a digraph and u a capacity vector for D. We say that an arc a of D is ***null*** if $u_a = 0$. It may seem that a null arc a can be simply deleted, but this is not so. If S is a source of D then S is also a source of $D - a$, but a source of $D - a$ may not be a source of D. In other words, the digraph $D - a$ may have cuts not present in D. (The sets of cuts of D and $D - a$ are the same if and only if there is a path in $D - a$ from the positive end of a to the negative end.)

In a capacitated digraph, the idea of a *set* of joins must be replaced with that of a *collection* of joins, where a ***collection*** is a "set" that may contain two or more copies of some elements, each copy contributing 1 to the size of the collection.

The definition of packing must be updated to take into account the capacities of the arcs. Thus, in a capacitated digraph (D, u), a collection $\mathcal{P}$ of joins is ***disjoint*** if

$$|\mathcal{P}(a)| \leq u_a$$

for every arc a, where $\mathcal{P}(a)$ denotes the collection of all the elements of $\mathcal{P}$ that contain a. In particular, $\mathcal{P}(a) = \emptyset$ whenever $u_a = 0$. To make u explicit, we may say that $\mathcal{P}$ is u***-disjoint***.

In a capacitated digraph (D, u), a ***packing of joins*** is any u-disjoint collection of joins of D. To make u explicit, we may say that $\mathcal{P}$ is a u***-packing***. Lemma 4.1 can be generalized as follows:

Lemma 5.1 (basic inequality) *In any capacitated digraph (D, u), for any cut C and any packing $\mathcal{P}$ of joins, the inequality $|\mathcal{P}| \leq u(C)$ holds. Moreover, if $|\mathcal{P}| = u(C)$ then $|J \cap C| = 1$ for each J in $\mathcal{P}$ and $|\mathcal{P}(a)| = u_a$ for each a in C.*

PROOF: Let $\mathcal{P}$ a packing of joins and C a cut of D. For each element J of $\mathcal{P}$ there is an arc a of C such that $\mathcal{P}(a) \ni J$. Hence,

$$|\mathcal{P}| \leq \textstyle\sum_{a \in C} |\mathcal{P}(a)| \leq \sum_{a \in C} u_a = u(C). \tag{1}$$

Now suppose that $|\mathcal{P}| = u(C)$. Then the first "$\leq$" in (1) holds as "$=$" and therefore $|J \cap C| = 1$ for each J in $\mathcal{P}$. The second "$\leq$" also holds as "$=$", whence $|\mathcal{P}(a)| = u_a$ for each a in C. □

Edmonds and Giles proposed [EG77] the following generalization of Woodall's conjecture 1:

Conjecture 2 (Edmonds–Giles) *Every capacitated digraph (D, u) has a packing $\mathcal{P}$ of joins and a cut C such that $|\mathcal{P}| = u(C)$.*

DRAFT

Schrijver has shown [Sch80] that the Edmonds–Giles conjecture is false. (See more in Feofiloff [Feo05].) As we show in section 6, however, every K4-free digraph has a packing $\mathcal{P}$ and a cut C such that $|\mathcal{P}| = u(C)$.

EXAMPLE: Let G be the graph consisting of a circuit (y, r, z, s, y) and a path (y, t, z) such that t is distinct from r and s. Let D be an orientation of G such that r and s are sinks and t is a source. Let u be the capacity vector defined by the following table:

yr	zr	ys	zs	ty	tz
1	2	5	1	2	2

Let $J_1 := \{yr, zs, ty\}$, $J_2 := \{zr, ys, tz\}$, and $C := \{yr, zr\}$. Then C is a cut and $\mathcal{P} := \{J_1, J_2, J_2\}$ is a u-packing of joins. Of course $|\mathcal{P}| = 3 = u(C)$.

In a capacitated digraph (D, u), a cut C is ***minimum*** if there is no cut C' such that $u(C') < u(C)$. To make u explicit, we may say that the cut is u-***minimum***. The capacity of any u-minimum cut of D is denoted by

$$\tau(D, u).$$

If D has no cuts then $\tau(D, u) = \infty$; otherwise, $\tau(D, u)$ is finite. The size of a maximum u-packing of joins of D is denoted by

$$\nu(D, u).$$

According to lemma 5.1, every capacitated digraph (D, u) satisfies the inequality $\nu(D, u) \leq \tau(D, u)$. The Edmonds–Giles conjecture 2 can be stated as follows: $\nu(D, u) = \tau(D, u)$ for every capacitated digraph (D, u).

The ***transpose*** of a capacitated digraph (D, u) is the capacitated digraph $(\tilde{D}, \tilde{u})$ where $\tilde{D}$ is the transpose of D and $\tilde{u}$ is the capacity vector defined in the obvious way: $\tilde{u}_{wv} = u_{vw}$ for every arc wv of $\tilde{D}$. Clearly, $\nu(\tilde{D}, \tilde{u}) = \nu(D, u)$ and $\tau(\tilde{D}, \tilde{u}) = \tau(D, u)$.

Acyclic circuits satisfy the Edmonds–Giles conjecture, as the following observation shows:

Proposition 5.1 Let D be an acyclic digraph consisting of a single circuit. Then $\nu(D, u) = \tau(D, u)$ for any capacity vector u.

PROOF: Let J_1 be the set of forward-directed arcs and J_2 the set of backward-directed arcs of the circuit. Since D is acyclic, J_1 and J_2 are nonempty. Clearly, J_1 and J_2 are joins. Let a be an arc that minimizes u_a in J_1 and b an arc that minimizes u_b in J_2. Then $\tau(D, u) = u_a + u_b$. On the other hand, the collection consisting of u_a copies of J_1 and u_b copies of J_2 is u-disjoint. Hence, $\nu(D, u) = \tau(D, u)$. $\square$

5.1 Critical capacity and oriented triplets

An arc of a capacitated digraph is ***critical*** if it belongs to a minimum cut or it is null. In other words, an arc c of a capacitated digraph (D, u) is critical if $u_c = 0$ or there exists a u-minimum cut C such that $C \ni c$. If every arc of D is critical, we say that u is ***critical***.

If u is critical then, according to lemma 3.1, each arc of every directed circuit in D is null. Here is another simple property:

Property 5.1 *If u is critical then $u_a \leq \tau(D, u)$ for every arc a of D.*

DRAFT

PROOF: Suppose u is critical and let a be an arc of D. If a is null then of course $u_a \le \tau(D,u)$. If a is nonnull then it belongs to some minimum cut C and therefore $u_a \le u(C) = \tau(D,u)$. □

Suppose that the graph underlying D has a triplet. As the edges of the graph become arcs of D, each triplet becomes serial or alternating. A triplet (y,r,z) is ***serial*** if r is the negative end of one of its legs and the positive end of the other leg. Otherwise, the triplet is ***alternating***. In the latter case, r is a source or a sink and we say that $\partial\{r\}$ is the ***head cut*** of the triplet. If a triplet (y,r,z) is closed then exactly one of the pairs yz and zy is an arc of D; this arc is the ***base*** of the triplet.

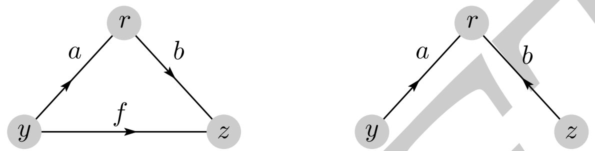

Figure 1: A serial closed triplet and an alternating open triplet.

Property 5.2 *If u is critical then $u_a = u_b$ for every serial triplet of D with legs a and b.*

PROOF: Suppose u is critical. If a and b are null then, of course, $u_a = u_b$. Now suppose that $u_a > 0$. Then a belongs to some minimum cut C. Since $C - a + b$ is also a cut, we have $\tau(D,u) \le u(C-a+b) = u(C) - u_a + u_b = \tau(D,u) - u_a + u_b$, whence $u_b \ge u_a$. In particular, $u_b > 0$, and so b belongs to some minimum cut B. Since $B - b + a$ is also a cut, we have $\tau(D,u) \le u(B-b+a) = u(B) - u_b + u_a = \tau(D,u) - u_b + u_a$, whence $u_a \ge u_b$. We conclude that $u_a = u_b$. □

Property 5.3 *If u is critical then $u_f + u_a \le \tau(D,u)$ and $u_f + u_b \le \tau(D,u)$ for every serial closed triplet with legs a and b and base arc f.*

PROOF: Suppose u is critical. If f is null then $u_f + u_a \le \tau(D,u)$ by property 5.1. If $u_f > 0$ then f belongs to some minimum cut C. Of course $C \ni a$ or $C \ni b$. In the first case, $u_f + u_a \le u(C) = \tau(D,u)$; in the second, $u_f + u_b \le u(C) = \tau(D,u)$. Since $u_a = u_b$ by property 5.2, both inequalities hold. □

Property 5.4 *If u is critical then $u_f + u_a \le \tau(D,u)$ or $u_f + u_b \le \tau(D,u)$ for every alternating closed triplet with legs a and b and base arc f. The first inequality holds if f and a have the same positive end or the same negative end and the second inequality holds if f and b have the same positive or the same negative end.*

PROOF: Suppose u is critical. If f is null, both inequalities hold by property 5.1. Now suppose that $u_f > 0$. Then f belongs to some u-minimum cut C. Adjust notation, by interchanging (D,u) with its transpose $(\tilde{D},\tilde{u})$ if necessary, so that r is a sink. Then f has the same positive end as a or the same positive end as b. In the first case, $C \ni a$ and therefore $u_f + u_a \le u(C) = \tau(D,u)$. In the second case, $C \ni b$ and therefore $u_f + u_b \le u(C) = \tau(D,u)$. □

An alternating triplet is ***special*** if there is a minimum cut that separates one of the feet of the triplet from the head and the other foot. In other words, an alternating triplet (y,r,z) is special if some minimum cut separates y from $\{r,z\}$ or separates z from $\{r,y\}$.

DRAFT

Property 5.5 *If u is critical then, any closed alternating triplet with a nonnull base is special.*

PROOF: Suppose u is critical and let (y, r, z) be a closed alternating triplet. Let f be the base of the triplet and suppose $u_f > 0$. Then f belongs to a minimum cut. Such cut necessarily separates y from $\{r, z\}$ or separates z from $\{y, r\}$. Hence, the triplet is special. □

Property 5.6 *If u is critical then the head cut of every nonspecial alternating triplet is u-minimum.*

PROOF: Suppose u is critical. Let r be the head and a and b the legs of an alternating triplet. The head cut of the triplet is, of course, $\partial\{r\} = \{a, b\}$. If a and b are null then $u(\partial\{r\}) = 0 = \tau(D, u)$ and so $\partial\{r\}$ is minimum. Now suppose $u_a > 0$ or $u_b > 0$. Adjust notation so that the first alternative holds. Then a belongs to some minimum cut C. If the triplet is not special then C separates r from the two feet of the triplet, and therefore $C \supseteq \partial\{r\}$. Hence, $\partial\{r\}$ is minimum. □

Special alternating triplets occur naturally when two triplets overlap:

Property 5.7 *If u is critical then, in any pair of overlapping alternating triplets, at least one of the triplets is special.*

PROOF: Suppose u is critical and let (y, r, z) and (r, z, w) be overlapping alternating triplets. Suppose that the first triplet is not special. Then, by virtue of property 5.6, the cut $\partial\{r\}$ is minimum. This cut separates r from $\{z, w\}$, i.e., separates one of the feet of the second triplet from the head and the other foot of the triplet. Hence, the second triplet is special. □

6 Theorem for K4-free digraphs

Lee and Wakabayashi found [LW01] that the Edmonds–Giles conjecture 2 is true when restricted to orientations of K4-free graphs:

Theorem 1 (Lee–Wakabayashi) *The equality $\nu(D, u) = \tau(D, u)$ holds for every capacitated digraph (D, u) such that D is an orientation of a K4-free graph.*

The theorem can be easily reduced to the acyclic case by contracting every directed circuit of D to a single vertex. The theorem can be also reduced to the case where the underlying graph is 2-connected by dealing with each 2-connected component in separate and then gluing all the solutions together. Hence, to establish theorem 1 it is sufficient to prove the following lemma:

Lemma 6.1 (Lee–Wakabayashi) *The equality $\nu(D, u) = \tau(D, u)$ holds for any capacitated digraph (D, u) such that D is an acyclic orientation of a 2-connected K4-free graph.*

The proof of this lemma is the object of the next section.

DRAFT

7 Proof of the Lee–Wakabayashi lemma

We prove lemma 6.1 by induction. Let G be a 2-connected K4-free graph, let D an acyclic orientation of G, and let u be a capacity vector for D. We proceed to show that $\nu(D, u) = \tau(D, u)$.

Of course G has more than 2 vertices. If G has only 3 vertices then G consists of a circuit of length 3 and then $\nu(D, u) = \tau(D, u)$ by virtue of proposition 5.1. This is the basis of the induction.

Now suppose that G has more than 3 vertices. The induction step has six cases. In what follows, we discuss the cases at an "executive" level and relegate the technical details to the next subsection. To simplify the wording, we shall say that a digraph is 2-connected and K4-free if its underlying graph has these properties.

The first case of the induction step, which we refer to as case 0, deals with the instances in which u is not critical.

Case 0: the capacity vector u is not critical. Let c be a nonnull arc that does not belong to a u-minimum cut. Define a new capacity vector u' for D as follows: u' coincides with u on all arcs except c and $u'_c = u_c - 1$. Of course $u'(C) = u(C) - 1$ for every cut C of D that contains c and $u'(C) = u(C)$ for all the other cuts of D. Hence, $\tau(D, u') = \tau(D, u)$.

Now consider the joins. Since $u' < u$, we may assume, as induction hypothesis, that $\nu(D, u') = \tau(D, u')$. Hence, there is a u'-packing $\mathcal{P}'$ of joins of D such that $|\mathcal{P}'| = \tau(D, u') = \tau(D, u)$. Since $|\mathcal{P}'(a)| \leq u'_a \leq u_a$ for each arc a of D, the collection $\mathcal{P}'$ is a u-packing. The existence of such u-packing, together with lemma 5.1, show that $\nu(D, u) = \tau(D, u)$.

From now on, suppose that u is critical and assume, as induction hypothesis, that $\nu(D', u') = \tau(D', u')$ for every capacitated acyclic digraph (D', u') such that $V(D') \subset V(D)$ and D' is 2-connected and K4-free.

By lemma 2.1, graph G has a vertex of degree 2 and therefore a triplet. As the edges of G are replaced with arcs of D, each triplet becomes serial or alternating, as defined in subsection 5.1. The next two cases of the induction step deal with the instances in which D has a serial triplet.

Case 1: D has a serial open triplet. Let (y, r, z) be a serial open triplet. Adjust notation, by interchanging (D, u) with its transpose $(\tilde{D}, \tilde{u})$ if necessary, so that the legs of the triplet are yr and rz.

Let f denote the ordered pair yz and let $D' := (D - r) + f$. Since D is acyclic, so is D'. Since D has more than 3 vertices and is 2-connected, D' is also 2-connected. Since D is K4-free, so is D'. Since $V(D') \subset V(D)$, we may assume, by induction hypothesis, that $\nu(D', u') = \tau(D', u')$ for any capacity vector u'.

Let $a := yr$ and $b := rz$. Define u' so that it coincides with u on all arcs of $D - r$ and has $u'_f = u_a$. Then lemma 7.1 (see the next subsection) shows that $\nu(D, u) = \tau(D, u)$.

Case 2: D has a serial closed triplet. Let (y, r, z) be a serial closed triplet. Adjust notation, by interchanging (D, u) with $(\tilde{D}, \tilde{u})$ if necessary, so that the legs of the triplet are yr and rz. Since the triplet is closed and D is acyclic, yz is the base of the triplet.

DRAFT

Let $D' := D - r$. Since D has more than 3 vertices and is 2-connected, D' is also 2-connected. Since D is K4-free and acyclic, so is D'. Since $V(D') \subset V(D)$, we may assume, by induction hypothesis, that $\nu(D', u') = \tau(D', u')$ for any capacity vector u'.

Let $a := yr$, $b := rz$, and $f := yz$. Define u' so that it coincides with u on all arcs of $D - r$ except f and has $u'_f = u_f + u_a$. Then lemma 7.2 (see the next subsection) shows that $\nu(D, u) = \tau(D, u)$.

From now on, we assume that D has no serial triplets, i.e., that every triplet is alternating.

Case 3: D **has an alternating closed triplet.** Let (y, r, z) be an alternating closed triplet. Adjust notation, by interchanging (D, u) with $(\tilde{D}, \tilde{u})$ if necessary, so that r is a sink. Adjust the notation further, by interchanging (y, r, z) with (z, r, y) if necessary, so that the base of the triplet is yz.

Let $D' := D - r$. Since D is acyclic and K4-free, so is D'. Since D has more than 3 vertices and is 2-connected, D' is also 2-connected. Since $V(D') \subset V(D)$, we may assume, by induction hypothesis, that $\nu(D', u') = \tau(D', u')$ for any capacity vector u'.

Let $a := yr$, $b := zr$, and $f := yz$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except f and has $u'_f = u_f + u_a$. Then lemma 7.3 shows that $\nu(D, u) = \tau(D, u)$.

From now on, we assume that D has no closed alternating triplets. Hence, every triplet is alternating and open. The next case of the induction step deals with the instances in which some alternating triplet is special, as defined in subsection 5.1.

Case 4: D **has a special alternating open triplet.** Let (y, r, z) be a special alternating open triplet. Adjust notation, by interchanging (D, u) with $(\tilde{D}, \tilde{u})$ if necessary, so that r is a sink. Adjust notation further, by interchanging (y, r, z) with (z, r, y) if necessary, so that a u-minimum cut C separates y from $\{r, z\}$.

Let f be the ordered pair yz and let $D' := (D - r) + f$. Due to the presence of C, there is no directed path from z to y in D and so D' is acyclic. Since D is K4-free, so is D'. Since D has more than 3 vertices and is 2-connected, D' is also 2-connected. Since $V(D') \subset V(D)$, we may assume, by induction hypothesis, that $\nu(D', u') = \tau(D', u')$ for any capacity vector u'.

Let $a := yr$ and $b := zr$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ and has $u'_f = u_a$. Then lemma 7.4 shows that $\nu(D, u) = \tau(D, u)$.

We assume, from now on, that every triplet of D is alternating, open, and nonspecial. In view of property 5.7, D has no overlapping triplets.

According to lemma 2.2, G has two ears, say P and Q, with a common origin and a common terminus. Each ear consists of a sequence of triplets, one triplet for each internal vertex of the ear. Since all triplets of D are alternating and do not overlap, P and Q have at most one internal vertex each. Since all triplets are open, P and Q have exactly one internal vertex each and therefore both P and Q consist of a single triplet. Of course these two triplets have the same pair of feet.

Case 5: D **has a pair of nonspecial alternating open triplets with common feet.** Let (y, r, z) and (y, s, z) be two nonspecial alternating open triplets. In view of property 5.6,

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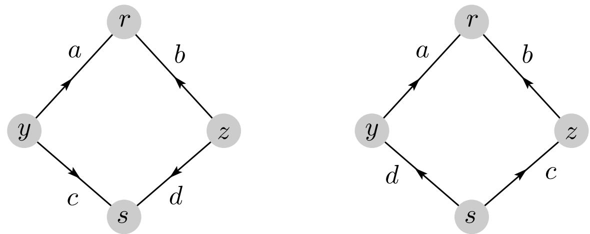

Figure 2: Two pairs of alternating open triplets with common feet.

the head cuts $\partial\{r\}$ and $\partial\{s\}$ are minimum. Adjust notation, by interchanging (D, u) with $(\tilde{D}, \tilde{u})$ if necessary, so that r is a sink.

Suppose first D has only 4 vertices. Then D is the circuit and so $\nu(D, u) = \tau(D, u)$ as proposition 5.1 shows.

From now on, we assume that D has more than 4 vertices. Let $D' := D - r$. Since D has more than 4 vertices and D is 2-connected, thus D' is 2-connected. Since D is K4-free and acyclic, so is D'. Since $V(D') \subset V(D)$, we may assume, by induction hypothesis, that $\nu(D', u') = \tau(D', u')$ for any u'.

Suppose first that s is a sink. Let $a := yr$, $b := zr$, $c := ys$, and $d := zs$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except c and d and has $u'_c = u_c + u_a$ and $u'_d = u_d + u_b$. Then lemma 7.5 shows that $\nu(D, u) = \tau(D, u)$.

Now suppose that s is a source. Let $a := yr$, $b := zr$, $d := sy$, and $c := sz$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except d and c and has $u'_d = u_d + u_a$ and $u'_c = u_c + u_b$. Then lemma 7.6 shows that $\nu(D, u) = \tau(D, u)$.

The proof of lemma 6.1 has now been reduced to lemmas 7.1 to 7.6. These lemmas will be discussed in the next subsection.

7.1 Main lemmas

Lemmas 7.1 to 7.6 contain the gist of the proof of the Lee–Wakabayashi lemma. Each lemma features a digraph D and a digraph D' derived from D. We shall write

$$\partial \quad \text{and} \quad \partial'$$

to indicate the cuts associated with the sources and sinks of D and D' respectively.

Lemma 7.1 (serial open triplet) *Let (y, r, z) be a serial open triplet in a capacitated digraph (D, u) such that u is critical. Suppose $a := yr$ and $b := rz$ are the arcs of the triplet. Let f denote the ordered pair yz and let $D' := (D - r) + f$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ and has $u'_f = u_a$. If $\nu(D', u') = \tau(D', u')$ then $\nu(D, u) = \tau(D, u)$.*

PROOF: Let $\mathcal{P}'$ be a maximum u'-packing of joins of D'. Then $|\mathcal{P}'| = \tau(D', u')$. Proposition 7.1 (see next subsection) shows that $\tau(D', u') \geq \tau(D, u)$. Adjust notation, by discarding some elements of $\mathcal{P}'$ is necessary, so that

$$|\mathcal{P}'| = \tau(D, u).$$

DRAFT

Let $\mathcal{Q}' := \mathcal{P}'(f)$ and $\mathcal{S}' := \mathcal{P}' \setminus \mathcal{P}'(f)$. Clearly, $|\mathcal{Q}'| \leq u'_f = u_a$ since $\mathcal{P}'$ is u'-disjoint. Proposition 7.7 (see next subsection) shows that, for each J' in $\mathcal{Q}' \cup \mathcal{S}'$,

if $J' \in \mathcal{Q}'$ then $J' - f + a + b$ is a join of D, and
if $J' \in \mathcal{S}'$ then J' is a join of D.

Let $\mathcal{Q}$ be the collection of all sets of the form $J' - f + a + b$ with J' in $\mathcal{Q}'$ and let $\mathcal{S} := \mathcal{S}'$. Of course $\mathcal{P} := \mathcal{Q} \cup \mathcal{S}$ is a collection of joins of D. We argue next that $\mathcal{P}$ is u-disjoint. For arc a, we have

$$|\mathcal{P}(a)| = |\mathcal{Q}(a)| + |\mathcal{S}(a)| = |\mathcal{Q}'| + 0 \leq u_a.$$

An analogous argument shows that $|\mathcal{P}(b)| \leq u_b$ since $u_b = u_a$ by property 5.2. Moreover, for each arc e of $D - a - b$, we have $|\mathcal{P}(e)| = |\mathcal{P}'(e)| \leq u'_e = u_e$. We conclude that $\mathcal{P}$ is u-disjoint. Finally, since each element of $\mathcal{P}$ belongs to one and only one of $\mathcal{Q}$ and $\mathcal{S}$, we have

$$|\mathcal{P}| = |\mathcal{Q}| + |\mathcal{S}| = |\mathcal{P}'| = \tau(D, u).$$

Hence $\nu(D, u) \geq |\mathcal{P}| = \tau(D, u)$ and therefore $\nu(D, u) = \tau(D, u)$. □

Lemma 7.2 (serial closed triplet) *Let (y, r, z) be a serial closed triplet in a capacitated digraph (D, u) such that u is critical. Suppose $a := yr$, $b := rz$, and $f := yz$ are the arcs of the triplet and let $D' := D - r$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except f and has $u'_f = u_f + u_a$. If $\nu(D', u') = \tau(D', u')$ then $\nu(D, u) = \tau(D, u)$.*

Proof: Let $\mathcal{P}'$ be a maximum u'-packing of joins of D'. Then $|\mathcal{P}'| = \tau(D', u')$. Proposition 7.2 shows that $\tau(D', u') \geq \tau(D, u)$. Adjust notation, by discarding some elements of $\mathcal{P}'$ if necessary, so that

$$|\mathcal{P}'| = \tau(D, u).$$

By property 5.3, $|\mathcal{P}'| \geq u_f + u_a$. On the other hand, $|\mathcal{P}'(f)| \leq u'_f = u_f + u_a$ since $\mathcal{P}'$ is u'-disjoint. Hence, we can adjust the notation, by adding f to some elements of $\mathcal{P}' \setminus \mathcal{P}'(f)$ if necessary, so that

$$|\mathcal{P}'(f)| = u_f + u_a.$$

This equality leads to the following partition $(\mathcal{Q}', \mathcal{R}', \mathcal{S}')$ of $\mathcal{P}'$:

$\mathcal{Q}'$ is any collection of u_a elements $\mathcal{P}'(f)$,
$\mathcal{R}'$ is the collection of the remaining u_f elements of $\mathcal{P}'(f)$, and
$\mathcal{S}'$ is the collection $\mathcal{P}' \setminus \mathcal{P}'(f)$.

Of course, $|\mathcal{Q}'| = u_a$ and $|\mathcal{R}'| = u_f$. Proposition 7.8 shows that, for every element J' of $\mathcal{Q}' \cup \mathcal{R}' \cup \mathcal{S}'$,

$J' - f + a + b$ and J' are joins of D.

Let $\mathcal{Q}$ be the collection of all sets of the form $J' - f + a + b$ for J' in $\mathcal{Q}'$. Let $\mathcal{R} := \mathcal{R}'$ and let $\mathcal{S} := \mathcal{S}'$. Of course $\mathcal{P} := \mathcal{Q} \cup \mathcal{R} \cup \mathcal{S}$ is a collection of joins of D. We argue next that $\mathcal{P}$ is u-disjoint. For arc a, we have

$$|\mathcal{P}(a)| = |\mathcal{Q}(a)| + |\mathcal{R}(a)| + |\mathcal{S}(a)| = |\mathcal{Q}'| + 0 + 0 = u_a.$$

Similarly, $|\mathcal{P}(b)| = u_b$ since $u_b = u_a$ by property 5.2. For arc f, we have

$$|\mathcal{P}(f)| = |\mathcal{Q}(f)| + |\mathcal{R}(f)| + |\mathcal{S}(f)| = 0 + |\mathcal{R}'| + 0 = u_f.$$

For each arc e of $D - a - b - f$, we have $|\mathcal{P}(e)| = |\mathcal{P}'(e)| \le u'_e = u_e$. We conclude that $\mathcal{P}$ is u-disjoint. Finally, since each element of $\mathcal{P}$ belongs to one and only one of $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$, we have

$$|\mathcal{P}| = |\mathcal{Q}| + |\mathcal{R}| + |\mathcal{S}| = |\mathcal{P}'| = \tau(D, u).$$

Hence, $\nu(D, u) \ge |\mathcal{P}| = \tau(D, u)$ and so $\nu(D, u) = \tau(D, u)$. □

Lemma 7.3 (alternating closed triplet) *Let (y, r, z) be an alternating closed triplet in a capacitated digraph (D, u) such that u is critical. Suppose r is a sink and yz is the base of the triplet. Let $a := yr$, $b := zr$, $f := yz$, and let $D' := D - r$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except f and has $u'_f = u_f + u_a$. If $\nu(D', u') = \tau(D', u')$ then $\nu(D, u) = \tau(D, u)$.*

Proof: Let $\mathcal{P}'$ be a maximum u'-packing of joins of D'. Then $|\mathcal{P}'| = \tau(D', u')$. Proposition 7.3 shows that $\tau(D', u') \ge \tau(D, u)$. Adjust notation, by discarding some elements of $\mathcal{P}'$ if necessary, so that

$$|\mathcal{P}'| = \tau(D, u).$$

By property 5.4, $|\mathcal{P}'| \ge u_f + u_a$. On the other had, $|\mathcal{P}'(f)| \le u'_f = u_f + u_a$ since $\mathcal{P}'$ is u'-disjoint. Hence, we can adjust the notation, by adding f to some elements of $\mathcal{P}' \setminus \mathcal{P}'(f)$ if necessary, so that

$$|\mathcal{P}'(f)| = u_f + u_a.$$

This equality leads to the following partition $(\mathcal{Q}', \mathcal{R}', \mathcal{S}')$ of $\mathcal{P}'$:

$\mathcal{Q}'$ is any collection of u_a elements of $\mathcal{P}'(f)$,
$\mathcal{R}'$ is the collection of the remaining u_f elements $\mathcal{P}'(f)$, and
$\mathcal{S}'$ is the collection $\mathcal{P}' \setminus \mathcal{P}'(f)$.

Clearly, $|\mathcal{Q}'| = u_a$, $|\mathcal{R}'| = u_f$, and

$$|\mathcal{S}'| = |\mathcal{P}'| - |\mathcal{P}'(f)| = \tau(D, u) - (u_f + u_a) \le (u_a + u_b) - (u_f + u_a) = u_b - u_f,$$

where "$\le$" holds because $\tau(D, u) \le u(\partial\{r\}) = u_a + u_b$. Proposition 7.9 shows that, for each element J' of $\mathcal{Q}' \cup \mathcal{R}' \cup \mathcal{S}'$,

$J' - f + a$ and $J' + b$ are joins of D.

Let $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ be the collections of joins of D defined as follows:

$\mathcal{Q}$ is the result of replacing each element J' of $\mathcal{Q}'$ with $J' - f + a$,
$\mathcal{R}$ is the result of replacing each element J' of $\mathcal{R}'$ with $J' + b$, and
$\mathcal{S}$ is the result of replacing each element J' of $\mathcal{S}'$ with $J' + b$.

Of course $\mathcal{P} := \mathcal{Q} \cup \mathcal{R} \cup \mathcal{S}$ is a collection of joins of D. We argue next that $\mathcal{P}$ is u-disjoint. For arc a, we have

$$|\mathcal{P}(a)| = |\mathcal{Q}(a)| + |\mathcal{R}(a)| + |\mathcal{S}(a)| = |\mathcal{Q}'| + 0 + 0 = u_a.$$

For arc b, we have

$$|\mathcal{P}(b)| = |\mathcal{Q}(b)| + |\mathcal{R}(b)| + |\mathcal{S}(b)| = 0 + |\mathcal{R}'| + |\mathcal{S}'| \le u_f + (u_b - u_f) = u_b.$$

For arc f, we have

$$|\mathcal{P}(f)| = |\mathcal{Q}(f)| + |\mathcal{R}(f)| + |\mathcal{S}(f)| = 0 + |\mathcal{R}'| + 0 = u_f.$$

DRAFT

Finally, for each arc e of $D - a - b - f$, we have $|\mathcal{P}(e)| = |\mathcal{P}'(e)| \le u'_e = u_e$. We conclude that $\mathcal{P}$ is u-disjoint. Finally, since each element of $\mathcal{P}$ belongs to one and only one of $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$, we have

$$|\mathcal{P}| = |\mathcal{Q}| + |\mathcal{R}| + |\mathcal{S}| = |\mathcal{P}'| = \tau(D, u).$$

Hence $\nu(D, u) \ge |\mathcal{P}| = \tau(D, u)$ and so $\nu(D, u) = \tau(D, u)$. □

Lemma 7.4 (special alternating open triplet) *Let (y, r, z) be an alternating open triplet in a capacitated digraph (D, u). Suppose that D is 2-connected and r is a sink. Let $a := yr$ and $b := zr$. Let f be the ordered pair yz and let $D' := (D - r) + f$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ and has $u'_f = u_a$. Suppose that some u-minimum cut of D separates y from $\{r, z\}$. If $\nu(D', u') = \tau(D', u')$ then $\nu(D, u) = \tau(D, u)$.*

Proof: Let $\mathcal{P}'$ be a maximum u'-packing of joins of D'. Then $|\mathcal{P}'| = \tau(D', u')$. Proposition 7.4 shows that $\tau(D', u') \ge \tau(D, u)$. Actually, equality holds, as we argue next.

Let Y be a nontrivial source of D such that $Y \cap \{y, r, z\} = \{y\}$ and $u(\partial Y) = \tau(D, u)$. Clearly, Y is also a nontrivial source of D' and $\partial' Y - f = \partial Y - a$. Since $u'_f = u_a$, we have

$$\tau(D, u) \;=\; u(\partial Y) \;=\; u'(\partial' Y) \;\ge\; \tau(D', u') \;\ge\; \tau(D, u)$$

and therefore $\tau(D', u') = \tau(D, u)$. Hence,

$$|\mathcal{P}'| = \tau(D, u) \tag{2}$$

and $|\mathcal{P}'| = u'(\partial' Y)$. Since $\partial' Y$ contains f, the second part of lemma 5.1, with $(D', u', \mathcal{P}', f, \partial' Y)$ in place of $(D, u, \mathcal{P}, a, C)$, implies that

$$|\mathcal{P}'(f)| = u'_f. \tag{3}$$

Let $\mathcal{Q}' := \mathcal{P}'(f)$ and $\mathcal{S}' := \mathcal{P}' \setminus \mathcal{P}'(f)$. Clearly, $|\mathcal{Q}'| = |\mathcal{P}'(f)| = u'_f = u_a$ due to (3) and $|\mathcal{S}'| = |\mathcal{P}'| - |\mathcal{Q}'| = \tau(D, u) - u_a \le (u_a + u_b) - u_a = u_b$, where the inequality holds because $\{a, b\}$ is a cut of D.

There is no harm in assuming that each join J' in $\mathcal{Q}'$ is minimal, and therefore that $J' - f$ is not a join of D'. With this assumption, and the 2-connectedness of D, propositions 7.10 and 7.11 apply and show that, for each J' in $\mathcal{Q}' \cup \mathcal{S}'$,

if $J' \in \mathcal{Q}'$ then $J' - f + a$ is a join of D, and
if $J' \in \mathcal{S}'$ then $J' + b$ is a join of D.

Let $\mathcal{Q}$ and $\mathcal{S}$ be the collections of joins of D defined as follows:

$\mathcal{Q}$ is the result of replacing each element J' of $\mathcal{Q}'$ with $J' - f + a$ and
$\mathcal{S}$ is the result of replacing each element J' of $\mathcal{S}'$ with $J' + b$.

Of course $\mathcal{P} := \mathcal{Q} \cup \mathcal{S}$ is a collection of joins of D. Next, we argue that $\mathcal{P}$ is u-disjoint. For arc a, we have

$$|\mathcal{P}(a)| = |\mathcal{Q}(a)| + |\mathcal{S}(a)| = |\mathcal{Q}'| + 0 = u_a.$$

For arc b, we have

$$|\mathcal{P}(b)| = |\mathcal{Q}(b)| + |\mathcal{S}(b)| = 0 + |\mathcal{S}'| \le u_b.$$

For any arc e of $D - a - b$, we have $|\mathcal{P}(e)| = |\mathcal{P}'(e)| \le u'_e = u_e$. Hence, $\mathcal{P}$ is u-disjoint. Finally, since each element of $\mathcal{P}$ belongs to one and only one of $\mathcal{Q}$ and $\mathcal{S}$,

$$|\mathcal{P}| = |\mathcal{Q}| + |\mathcal{S}| = |\mathcal{P}'| = \tau(D, u)$$

DRAFT

due to (2). Hence $\nu(D,u) \ge |\mathcal{P}| = \tau(D,u)$ and so $\nu(D,u) = \tau(D,u)$. □

Lemma 7.5 (pair of nonspecial alternating open triplets) *Let (y,r,z) and (y,s,z) be two alternating open triplets in a capacitated digraph (D,u). Suppose r and s are sinks and the cuts $\partial\{r\}$ and $\partial\{s\}$ are u-minimum. Let $a := yr$, $b := zr$, $c := ys$, $d := zs$ and let $D' := D - r$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except c and d and has $u'_c = u_c + u_a$ and $u'_d = u_d + u_b$. If $\nu(D',u') = \tau(D',u')$ then $\nu(D,u) = \tau(D,u)$.*

PROOF: Let $\mathcal{P}'$ be a maximum u'-packing of joins of D'. Then $|\mathcal{P}'| = \tau(D',u')$. Proposition 7.5 shows that $\tau(D',u') \ge \tau(D,u)$. Adjust the notation, by discarding some elements of $\mathcal{P}'$ if necessary, so that

$$|\mathcal{P}'| = \tau(D,u).$$

Clearly $\partial\{r\} = \{a,b\}$ and $\partial\{s\} = \{c,d\}$. Since the cuts $\partial\{r\}$ and $\partial\{r\}$ are minimum, thus

$$u_a + u_b = u_c + u_d = \tau(D,u). \tag{4}$$

As a consequence, $u_a + u_c \ge \tau(D,u)$ if and only if $u_b + u_d \le \tau(D,u)$. Adjust the notation, by interchanging (y,r,z) with (z,r,y) and (y,s,z) with (z,s,y) if necessary, so that

$$u_b + u_d \le \tau(D,u). \tag{5}$$

Since $\mathcal{P}'$ is u'-disjoint, $|\mathcal{P}'(d)| \le u'_d = u_d + u_b$. On the other hand, $|\mathcal{P}'| = \tau(D,u) \ge u_d + u_b$ due to (5). Hence, we can adjust the notation, by adding d to some elements of $\mathcal{P}' \smallsetminus \mathcal{P}'(d)$ if necessary, so that

$$|\mathcal{P}'(d)| = u_d + u_b. \tag{6}$$

Equality (6) leads to the following partition $(\mathcal{Q}', \mathcal{R}', \mathcal{S}')$ of $\mathcal{P}'$:

$\mathcal{Q}'$ is any collection of u_b elements of $\mathcal{P}'(d)$,
$\mathcal{R}'$ is the collection of the remaining u_d elements of $\mathcal{P}'(d)$, and
$\mathcal{S}'$ is the collection $\mathcal{P}' \smallsetminus \mathcal{P}'(d)$.

Of course $|\mathcal{Q}'| = u_b$, $|\mathcal{R}'| = u_d$, and $|\mathcal{S}'| = |\mathcal{P}'| - |\mathcal{P}'(d)| = \tau(D,u) - (u_d + u_b) = (u_a + u_b) - (u_d + u_b)$ due to (6) and (4) and therefore

$$|\mathcal{S}'| = u_a - u_d = u_c - u_b.$$

Proposition 7.12 shows that, for each J' in $\mathcal{Q}' \cup \mathcal{R}' \cup \mathcal{S}'$,

$J' + b - d + c$, $J' + a - c + d$, and $J' + a$ are joins of D.

Let $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ be the collections of joins of D defined as follows:

$\mathcal{Q}$ is the result of replacing each element J' of $\mathcal{Q}'$ with $J' + b - d + c$,
$\mathcal{R}$ is the result of replacing each element J' of $\mathcal{R}'$ with $J' + a - c + d = J' + a - c$, and
$\mathcal{S}$ is the result of replacing each element J' of $\mathcal{S}'$ with $J' + a$.

Of course $\mathcal{P} := \mathcal{Q} \cup \mathcal{R} \cup \mathcal{S}$ is a collection of joins of D. We argue next that $\mathcal{P}$ is u-disjoint. For arc a, we have

$$|\mathcal{P}(a)| = |\mathcal{Q}(a)| + |\mathcal{R}(a)| + |\mathcal{S}(a)| = 0 + |\mathcal{R}'| + |\mathcal{S}'| = u_d + (u_a - u_d) = u_a.$$

For arc b, we have

$$|\mathcal{P}(b)| = |\mathcal{Q}(b)| + |\mathcal{R}(b)| + |\mathcal{S}(b)| = |\mathcal{Q}'| + 0 + 0 = u_b.$$

DRAFT

For arc c, we have

$$|\mathcal{P}(c)| = |\mathcal{Q}(c)| + |\mathcal{R}(c)| + |\mathcal{S}(c)| \le |\mathcal{Q}'| + 0 + |\mathcal{S}'| = u_b + (u_c - u_b) = u_c.$$

For arc d, we have

$$|\mathcal{P}(d)| = |\mathcal{Q}(d)| + |\mathcal{R}(d)| + |\mathcal{S}(d)| = 0 + |\mathcal{R}'| + 0 = u_d.$$

For any arc e of $D-a-b-c-d$, we have $|\mathcal{P}(e)| = |\mathcal{P}'(e)| \le u'_e = u_e$. Hence, $\mathcal{P}$ is u-disjoint. Finally, since each element of $\mathcal{P}$ belongs to one and only one of $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$, we have

$$|\mathcal{P}| = |\mathcal{Q}| + |\mathcal{R}| + |\mathcal{S}| = |\mathcal{P}'| = \tau(D, u).$$

Hence $\nu(D, u) \ge |\mathcal{P}| = \tau(D, u)$ and so $\nu(D, u) = \tau(D, u)$. □

Lemma 7.6 (pair of nonspecial alternating open triplets) *Let (y, r, z) and (y, s, z) be two alternating open triplets in a capacitated digraph (D, u). Suppose r is a sink, s is a source and the cuts $\partial\{r\}$ and $\partial\{s\}$ are u-minimum. Let $a := yr$, $b := zr$, $d := sy$, $c := sz$, and let $D' := D - r$. Let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except c and d and has $u'_c = u_c + u_a$ and $u'_d = u_d + u_b$. If $\nu(D', u') = \tau(D', u')$ then $\nu(D, u) = \tau(D, u)$.*

PROOF: The proof of this lemma is formally identical to the proof of lemma 7.5 if the invocations of propositions 7.5 and 7.12 are replaced with the invocations of propositions 7.6 and 7.13 respectively. □

To complete the proofs of lemmas 7.1 to 7.6, we must verify propositions 7.1 to 7.13. This is done in the next subsection.

7.2 Auxiliary propositions

This subsection proves propositions 7.1 to 7.13. Most proofs are rather tedious and repetitive, the exception being the proofs of propositions 7.10, 7.12, and 7.13.

Each proposition features two digraphs, D and D'. As in the previous subsection, we write

$$\partial \quad \text{and} \quad \partial'$$

to indicate the cuts associated with sources and sinks of D and D' respectively.

Proposition 7.1 (cuts for serial open triplet) Let (y, r, z) be a serial triplet in a capacitated digraph (D, u). Suppose that $a := yr$ and $b := rz$ are the legs of the triplet. Let $f := yz$, let $D' := (D - r) + f$ and let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ and has $u'_f = u_a$. Then $\tau(D', u') \ge \tau(D, u)$.

PROOF: Let S' be a nontrivial source of D'. The intersection $S' \cap \{y, z\}$ has only three possible values. For each of these values we argue that $u'(\partial' S') = u(\partial S)$ for some nontrivial source S of D.

Case i: $S' \cap \{y, z\} = \emptyset$. Then S' is a nontrivial source of D and $\partial' S' = \partial S'$. Hence $u'(\partial' S') = u(\partial S')$.

DRAFT

Case ii: $S' \cap \{y, z\} = \{y, z\}$. Then $S' + r$ is a nontrivial source of D and $\partial' S' = \partial(S' + r)$. Hence $u'(\partial' S') = u(\partial(S' + r))$.

Case iii: $S' \cap \{y, z\} = \{y\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S' - a + f$. Hence, $u'(\partial' S') = u'(\partial S' - a + f) = u'(\partial S' - a) + u'_f = u(\partial S' - a) + u'_f = u(\partial S') - u_a + u'_f = u(\partial S')$. □

Proposition 7.2 (cuts for serial closed triplet) Let (y, r, z) be a serial triplet in a capacitated digraph (D, u). Suppose that $a := yr$ and $b := rz$ are the legs of the triplet. Let $f := yz$ be the base of the triplet, let $D' := D - r$. and let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except f and has $u'_f = u_f + u_a$. Then $\tau(D', u') \geq \tau(D, u)$.

PROOF: Let S' be a nontrivial source of D'. The intersection $S' \cap \{y, z\}$ has only three possible values. For each of these values we argue that $u'(\partial' S') = u(\partial S)$ for some nontrivial source S of D.

Case i: $S' \cap \{y, z\} = \emptyset$. Then S' is a nontrivial source of D and $\partial' S' = \partial S'$. Hence $u'(\partial' S') = u(\partial S')$.

Case ii: $S' \cap \{y, z\} = \{y, z\}$. Then $S' + r$ is a nontrivial source of D and $\partial' S' = \partial(S' + r)$. Hence $u'(\partial' S') = u(\partial(S' + r))$.

Case iii: $S' \cap \{y, z\} = \{y\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S' - a$. Hence $u'(\partial' S') = u'(\partial S' - a) = u'(\partial S' - a - f) + u'_f = u(\partial S' - a - f) + u'_f = u(\partial S') - u_a - u_f + u'_f = u(\partial S')$. □

Proposition 7.3 (cuts for alternating closed triplet) *Let (y, r, z) be an alternating closed triplet in a capacitated digraph (D, u). Suppose that $a := yr$ and $b := zr$ are the legs of the triplet. Let $f := yz$ be the base of the triplet, let $D' := D - r$, and let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ except f and has $u'_f = u_f + u_a$. Then $\tau(D', u') \geq \tau(D, u)$.*

PROOF: Let S' be a nontrivial source of D'. For each of the three possible values of the intersection $S' \cap \{y, z\}$, we argue next that $u'(\partial' S') = u(\partial S)$ for some nontrivial source S of D.

Case i: $S' \cap \{y, z\} = \emptyset$. Then S' is a nontrivial source of D and $\partial' S' = \partial S'$, whence $u'(\partial' S') = u(\partial S')$.

Case ii: $S' \cap \{y, z\} = \{y, z\}$. Then $S' + r$ is a nontrivial source of D and $\partial' S' = \partial(S' + r)$, whence $u'(\partial' S') = u(\partial(S' + r))$.

Case iii: $S' \cap \{y, z\} = \{y\}$. Then S' is a nontrivial source of D. Since the triplet is closed then $\partial' S' = \partial S' - a$ and therefore $u'(\partial' S') = u'(\partial S' - a) = u'(\partial S' - a - f) + u'_f = u(\partial S' - a - f) + u'_f = u(\partial S') - u_a - u_f + u'_f = u(\partial S')$. □

Proposition 7.4 (cuts for alternating open triplet) *Let (y, r, z) be an alternating open triplet in a capacitated digraph (D, u). Suppose that $a := yr$ and $b := zr$ are the legs of the triplet. Let f be the pair yz, let $D' := (D - r) + f$, and let u' be the capacity vector for D' that coincides with u on all arcs of $D - r$ and has $u'_f = u_a$. Then $\tau(D', u') \geq \tau(D, u)$.*

PROOF: Let S' be a nontrivial source of D'. For each of the three possible values of the intersection $S' \cap \{y, z\}$, we argue next that $u'(\partial' S') = u(\partial S)$ for some nontrivial source S

DRAFT

of D.

Case i: $S' \cap \{y, z\} = \emptyset$. Then S' is a nontrivial source of D and $\partial' S' = \partial S'$, whence $u'(\partial' S') = u(\partial S')$.

Case ii: $S' \cap \{y, z\} = \{y, z\}$. Then $S' + r$ is a nontrivial source of D and $\partial' S' = \partial(S' + r)$, whence $u'(\partial' S') = u(\partial(S' + r))$.

Case iii: $S' \cap \{y, z\} = \{y\}$. Then S' is a nontrivial source of D. Since the triplet is open then $\partial' S' = \partial S' - a + f$ and therefore $u'(\partial' S') = u'(\partial S' - a + f) = u'(\partial S' - a) + u'_f = u(\partial S' - a) + u'_f = u(\partial S') - u_a + u'_f = u(\partial S')$. $\square$

Proposition 7.5 (cuts for pair of alternating triplets) *Let* (y, r, z) *and* (y, s, z) *be two alternating triplets in a capacitated digraph* (D, u). *Suppose that* $a := yr$ *and* $b := zr$ *are the legs of the first triplet and* $c := ys$ *and* $d := zs$ *are the legs of the second triplet. Let* $D' := D - r$ *and let* u' *be the capacity vector for* D' *that coincides with* u *on all arcs of* $D - r$ *except* c *and* d *and has* $u'_c = u_c + u_a$ *and* $u'_d = u_d + u_b$. *Then* $\tau(D', u') \geq \tau(D, u)$.

PROOF: Let S' be a nontrivial source of D'. For each of the five possible values of the intersection $S' \cap \{y, s, z\}$, we show next that $u'(\partial' S') = u(\partial S)$ for some nontrivial source S of D.

Case i: $S' \cap \{y, s, z\} = \emptyset$. Then S' is a nontrivial source of D and $\partial S' = \partial' S'$. Hence $u'(\partial' S') = u(\partial S')$.

Case ii: $S' \cap \{y, s, z\} = \{y, s, z\}$. Then $S' + r$ is a nontrivial source of D and $\partial(S' + r) = \partial' S'$. Hence $u'(\partial' S') = u(\partial(S' + r))$.

Case iii: $S' \cap \{y, s, z\} = \{y\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S' - a$. Moreover, $c \in \partial S'$ but $d \notin \partial S'$, whence $u'(\partial' S') = u'(\partial S' - a) = u'(\partial S' - a - c) + u'_c = u(\partial S' - a - c) + u'_c = u(\partial S') - u_a - u_c + u'_c = u(\partial S')$.

Case iv: $S' \cap \{y, s, z\} = \{z\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S' - b$. Moreover, $d \in \partial S'$ but $c \notin \partial S'$, whence $u'(\partial' S') = u'(\partial S' - b) = u'(\partial S' - b - d) + u'_d = u(\partial S' - b - d) + u'_d = u(\partial S') - u_b - u_d + u'_d = u(\partial S')$.

Case v: $S' \cap \{y, s, z\} = \{y, z\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S' - a - b$. Moreover, $c \in \partial S'$ and $d \in \partial S'$, whence $u'(\partial' S') = u'(\partial S' - a - b) = u'(\partial S' - a - b - c - d) + u'_c + u'_d = u(\partial S' - a - b - c - d) + u'_c + u'_d = u(\partial S') - u_a - u_b - u_c - u_d + u'_c + u'_d = u(\partial S')$. $\square$

Proposition 7.6 (cuts for pair of alternating triplets) *Let* (y, r, z) *and* (y, s, z) *be two alternating triplets in a capacitated digraph* (D, u). *Suppose that* $a := yr$ *and* $b := zr$ *are the legs of the first triplet and* $d := sy$ *and* $c := sz$ *are the legs of the second triplet. Let* $D' := D - r$ *and let* u' *be the capacity vector for* D' *that coincides with* u *on all arcs of* $D - r$ *except* d *and* c *and has* $u'_d = u_d + u_b$ *and* $u'_c = u_c + u_a$. *Then* $\tau(D', u') \geq \tau(D, u)$.

PROOF: Let S' be a nontrivial source of D'. For each of the five possible values of the intersection $S' \cap \{y, s, z\}$, we show next that $u'(\partial' S') = u(\partial S)$ for some nontrivial source S of D.

Case i: $S' \cap \{y, s, z\} = \emptyset$. Then S' is a nontrivial source of D and $\partial' S' = \partial S'$. Hence $u'(\partial' S') = u(\partial S')$.

Case ii: $S' \cap \{y, s, z\} = \{y, s, z\}$. Then $S' + r$ is a nontrivial source of D and $\partial' S' = \partial(S' + r)$. Hence $u'(\partial' S') = u(\partial(S' + r))$.

DRAFT

Case iii: $S' \cap \{y, s, z\} = \{y, s\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S' - a$. Moreover, $c \in \partial S'$ but $d \notin \partial S'$, whence $u'(\partial' S') = u'(\partial S' - a) = u'(\partial S' - a - c) + u'_c = u(\partial S' - a - c) + u'_c = u(\partial S') - u_a - u_c + u'_c = u(\partial S')$.

Case iv: $S' \cap \{y, s, z\} = \{z, s\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S' - b$. Moreover, $d \in \partial S'$ but $c \notin \partial S'$, whence $u'(\partial' S') = u'(\partial S' - b) = u'(\partial S' - b - d) + u'_d = u(\partial S' - b - d) + u'_d = u(\partial S') - u_b - u_d + u'_d = u(\partial S')$.

Case v: $S' \cap \{y, s, z\} = \{s\}$. Then S' is a nontrivial source of D and $\partial' S' = \partial S'$. Hence $u'(\partial' S') = u(\partial S')$. □

Proposition 7.7 (joins for serial open triplet) *Let (y, r, z) be a serial open triplet in a digraph D. Suppose that $a := yr$ and $b := rz$ are the legs of the triplet. Let $f := yz$ and $D' := (D - r) + f$. For any join J' of D', if $f \in J'$ then $J' - f + a + b$ is a join of D and if $f \notin J'$ then J' is a join of D.*

PROOF: Let J' be a join of D' and let S be a nontrivial source of D. For each of the four possible values of the intersection $S \cap \{y, r, z\}$, we argue next that $(J' - f + a + b) \cap \partial S$ or $J' \cap \partial S$ is nonempty.

Case i: $S \cap \{y, r, z\} = \emptyset$. Then $\partial S = \partial' S$ and therefore $(J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J' - f + a + b) \cap \partial S \neq \emptyset$ and $J' \cap \partial S \neq \emptyset$.

Case ii: $S \cap \{y, r, z\} = \{y, r, z\}$. Then $\partial S = \partial'(S - r)$ and therefore $(J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. In particular, $(J' - f + a + b) \cap \partial S \neq \emptyset$ and $J' \cap \partial S \neq \emptyset$.

Case iii: $S \cap \{y, r, z\} = \{y\}$. Clearly $(J' - f + a + b) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$. In addition, $\partial S = \partial' S - f + a$ and therefore $J' \cap \partial S = J' \cap (\partial' S - f + a) = J' \cap (\partial' S + a)$ provided $f \notin J'$. Finally, $J' \cap (\partial' S + a) = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'.

Case iv: $S \cap \{y, r, z\} = \{y, r\}$. Clearly $(J' - f + a + b) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$. Now suppose that $f \notin J'$ and note that $\partial S = \partial'(S - r) - f + b$. Then $J' \cap \partial S = J' \cap (\partial'(S - r) - f + b) = J' \cap (\partial'(S - r) + b) = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. □

Proposition 7.8 (joins for serial closed triplet) *Let (y, r, z) be a serial closed triplet in a digraph D. Suppose that $a := yr$ and $b := rz$ are the legs of the triplet and $f := yz$ is the base of the triplet. Let $D' := D - r$. For any join J' of D', both $J' - f + a + b$ and J' are joins of D.*

PROOF: Let J' be a join of D' and let S be a nontrivial source of D. For each of the four possible values of the intersection $S \cap \{y, r, z\}$, we argue next that $(J' - f + a + b) \cap \partial S$ and $J' \cap \partial S$ are nonempty.

Case i: $S \cap \{y, r, z\} = \emptyset$. Then $\partial S = \partial' S$ and therefore $(J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J' - f + a + b) \cap \partial S \neq \emptyset$ and $J' \cap \partial S \neq \emptyset$.

Case ii: $S \cap \{y, r, z\} = \{y, r, z\}$. Then $\partial S = \partial'(S - r)$ and therefore $(J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. In particular, $(J' - f + a + b) \cap \partial S \neq \emptyset$ and $J' \cap \partial S \neq \emptyset$.

Case iii: $S \cap \{y, r, z\} = \{y\}$. Clearly $(J' - f + a + b) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$. In addition, $\partial S = \partial' S + a$ and therefore $J' \cap \partial S = J' \cap (\partial' S + a) = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'.

DRAFT

Case iv: $S \cap \{y, r, z\} = \{y, r\}$. Clearly $(J' - f + a + b) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$. In addition, $\partial S = \partial'(S - r) + b$ and therefore $J' \cap \partial S = J' \cap (\partial'(S - r) + b) = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. □

Proposition 7.9 (joins for alternating closed triplet) *Let (y, r, z) be an alternating closed triplet in a digraph D. Suppose that $a := yr$ and $b := zr$ are the legs and $f := yz$ is the base of the triplet. Let $D' := D - r$. For any join J' of D', both $J' - f + a$ and $J' + b$ are joins of D.*

PROOF: Let J' be a join of D' and let S be a nontrivial source of D. For each of the four possible values of the intersection $S \cap \{y, r, z\}$, we argue next that $(J' - f + a) \cap \partial S$ and $(J' + b) \cap \partial S$ are nonempty.

Case i: $S \cap \{y, r, z\} = \emptyset$. Then $\partial S = \partial' S$ and therefore $(J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J' - f + a) \cap \partial S \neq \emptyset$ and $(J' + b) \cap \partial S \neq \emptyset$.

Case ii: $S \cap \{y, r, z\} = \{y, r, z\}$. Then $\partial S = \partial'(S - r)$ and therefore $(J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. In particular, $(J' - f + a) \cap \partial S \neq \emptyset$ and $(J' + b) \cap \partial S \neq \emptyset$.

Case iii: $S \cap \{y, r, z\} = \{y\}$. Then $(J' - f + a) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$. In addition, $\partial S = \partial' S + a$ and therefore $(J' + b) \cap \partial S = J' \cap \partial S = J' \cap (\partial' S + a) = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'.

Case iv: $S \cap \{y, r, z\} = \{y, z\}$. Then $(J' - f + a) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$ and $(J' + b) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$. □

Proposition 7.10 (join for alternating open triplet) *Let (y, r, z) be an alternating open triplet in a* 2-connected *digraph D. Suppose that $a := yr$ and $b := zr$ are the legs of the triplet. Let $f := yz$ and let $D' := (D - r) + f$. For any* minimal *join J' of D', if $f \in J'$ then $J' - f + a$ is a join of D.*

PROOF: Let J' be a minimal join of D' such that $J' \ni f$. Let S be a nontrivial source of D. For each of the five possible values of the intersection $S \cap \{y, r, z\}$ we argue next that $(J' - f + a) \cap \partial S$ is nonempty.

Case i: $S \cap \{y, r, z\} = \emptyset$. Then $\partial S = \partial' S$ and therefore $(J' - f + a) \cap \partial S = (J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'.

Case ii: $S \cap \{y, r, z\} = \{y, r, z\}$. Then $\partial S = \partial'(S - r)$ and therefore $(J' - f + a) \cap \partial S = (J' - f) \cap \partial S = J' \cap \partial S = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'.

Case iii: $S \cap \{y, r, z\} = \{y\}$. Then $(J' - f + a) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$.

Case iv: $S \cap \{y, r, z\} = \{y, z\}$. Same argument as in case iii.

Case v: $S \cap \{y, r, z\} = \{z\}$. We will show that $(J' - f) \cap \partial S$ is nonempty. Unlike in the previous cases, S is a not a source of D', but $\partial' S = \partial S - b + f$. Moreover, since the join J' is minimal, there exists a nontrivial source Y of D' such that

$$J' \cap \partial' Y = \{f\}.$$

Of course $Y \cap \{y, z\} = \{y\}$. Clearly, Y is also a source of D and $\partial Y = \partial' Y - f + a$, whence

$$(J' - f) \cap \partial Y = (J' - f) \cap (\partial' Y - f + a) = (J' - f) \cap \partial' Y = \emptyset. \tag{7}$$

DRAFT

Since S and Y are sources of D, also $U := S \cup Y$ and $I := S \cap Y$ are sources of D. Moreover, $\partial U \cup \partial I = \partial S \cup \partial Y$ and $\partial U \cap \partial I = \partial S \cap \partial Y$. This leads to the modular relation

$$|\partial U| + |\partial I| = |\partial S| + |\partial Y|\,.$$

The equality is preserved when we take the intersection of each term with an arbitrary set of arcs. In particular,

$$|(J' - f) \cap \partial U| + |(J' - f) \cap \partial I| = |(J' - f) \cap \partial S| + |(J' - f) \cap \partial Y|\,.$$

Since $(J' - f) \cap \partial Y = \emptyset$ by virtue of (7), the modular relation is reduced to

$$|(J' - f) \cap \partial U| + |(J' - f) \cap \partial I| = |(J' - f) \cap \partial S|\,.$$

We shall prove next that either $(J' - f) \cap \partial U$ is nonempty or $(J' - f) \cap \partial I$ is nonempty. This will imply that $(J' - f) \cap \partial S$ is nonempty and thereby conclude the analysis of case v.

Subcase v.1: $\partial U \neq \{a, b\}$. Of course $\partial U \supset \{a, b\}$. The set U is not only a source of D but also a source of D', since f has both ends in U. Moreover, $\partial' U = \partial U - a - b$. Hence, U is nontrivial in D' and therefore $J' \cap \partial' U \neq \emptyset$. Finally, $(J' - f) \cap \partial U = (J' - f) \cap (\partial' U + a + b) = (J' - f) \cap \partial' U = J' \cap \partial' U \neq \emptyset$, as claimed.

Subcase v.2: $\partial U = \{a, b\}$. Let G be the graph underlying D. Since G is 2-connected, there are two internally disjoint paths from y to z in G. Hence, there is a path P from y to z in $G - r$. Of course, P can be seen as a path from y to z in $D - r$. All the vertices of P are in U since $\partial U = \partial\{r\}$. Since the origin of P is in $Y \setminus S$ while the terminus is in $S \setminus Y$, some arc vw of P has one end in Y and the other in $U \setminus Y$. Since Y is a source, $v \in Y$ and $w \in U \setminus Y$. Since $U \setminus Y \subseteq S$ and S is a source of D, thus $v \in S$. We conclude that $v \in I$ and $vw \in \partial I$. But I is also a source of D' and $\partial' I = \partial I$, whence I is nontrivial in D' and therefore $J' \cap \partial' I \neq \emptyset$. Finally, $(J' - f) \cap \partial I = (J' - f) \cap \partial' I = J' \cap \partial' I \neq \emptyset$, as claimed. □

Proposition 7.11 (join for alternating open triplet) *Let (y, r, z) be an alternating open triplet in a digraph D. Suppose that $a := yr$ and $b := zr$ are the legs of the triplet. Let $f := yz$ and $D' := (D - r) + f$. For any join J' of D', if $f \notin J'$ then $J' + b$ is a join of D.*

PROOF: Let J' be a join of D' such that $J' \not\ni f$. Let S be a nontrivial source of D. For each of the five possible values of the intersection $S \cap \{y, r, z\}$ we argue next that $(J' + b) \cap \partial S \neq \emptyset$.

<u>Case i:</u> $S \cap \{y, r, z\} = \emptyset$. Then $\partial S = \partial' S$ and therefore $(J' + b) \cap \partial S = J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'.

<u>Case ii:</u> $S \cap \{y, r, z\} = \{y, r, z\}$. Then $\partial S = \partial'(S - r)$ and therefore $(J' + b) \cap \partial S = J' \cap \partial S = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'.

<u>Case iii:</u> $S \cap \{y, r, z\} = \{y\}$. Then $\partial S = \partial' S - f + a$ and therefore $(J' + b) \cap \partial S = J' \cap \partial S = J' \cap (\partial' S - f + a) = J' \cap (\partial' S - f)$. Suppose $f \notin J'$. Then $J' \cap (\partial' S - f) = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'.

<u>Case iv:</u> $S \cap \{y, r, z\} = \{y, z\}$. Then $(J' + b) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$.

<u>Case v:</u> $S \cap \{y, r, z\} = \{z\}$. Same argument as in case iv. □

Proposition 7.12 (joins for pair of alternating open triplets) *Let (y, r, z) and (y, s, z) be alternating open triplets in a digraph D. Suppose that $a := yr$ and $b := zr$ are*

DRAFT

the legs of the first triplet and $c := ys$ and $d := zs$ are the legs of the second triplet. Let $D' := D - r$. For any join J' of D', both $J' + a$ and $J' + b$ as well as $J' + a - c + d$ and $J' + b - d + c$ are joins of D.

PROOF: Let J' be a join of D' and S a nontrivial source of D. For each of the seven possible values of the intersection $S \cap \{y, r, s, z\}$ we argue next that $J' + a$ and $J' + b$ as well as $(J' + a - c + d) \cap \partial S$ and $(J' + b - d + c) \cap \partial S$ are nonempty.

Case i: $S \cap \{y, r, s, z\} = \emptyset$. Then $\partial S = \partial' S$ and therefore $(J' - c - d) \cap \partial S = J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $J' + a$ and $J' + b$ as well as $(J' + a - c + d) \cap \partial S$ and $(J' + b - d + c) \cap \partial S$ are nonempty.

Case ii: $S \cap \{y, r, s, z\} = \{y, r, s, z\}$. Then $\partial S = \partial'(S - r)$ and therefore $(J' - c - d) \cap \partial S = J' \cap \partial S = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. In particular, $J' + a$ and $J' + b$ as well as $(J' + a - c + d) \cap \partial S$ and $(J' + b - d + c) \cap \partial S$ are nonempty.

Case iii: $S \cap \{y, r, s, z\} = \{y\}$. Then $\partial S = \partial' S + a$ and therefore $J' \cap \partial S = J' \cap (\partial' S + a) = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J' + a) \cap \partial S$ and $(J' + b) \cap \partial S$ are nonempty. In addition, $(J' + a - c + d) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$ and $(J' + b - d + c) \cap \partial S \supseteq \{c\} \cap \partial S \neq \emptyset$.

Case iv: $S \cap \{y, r, s, z\} = \{z\}$. Then $\partial S = \partial' S + b$ and therefore $J' \cap \partial S = J' \cap (\partial' S + b) = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J' + a) \cap \partial S$ and $(J' + b) \cap \partial S$ are nonempty. In addition, $(J' + a - c + d) \cap \partial S \supseteq \{d\} \cap \partial S \neq \emptyset$ and $(J' + b - d + c) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$.

Case v: $S \cap \{y, r, s, z\} = \{y, s, z\}$. Then $(J' + a) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$ and $(J' + b) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$. Similarly, $(J' + a - c + d) \cap \partial S \neq \emptyset$ and $(J' + b - d + c) \cap \partial S \neq \emptyset$.

Case vi: $S \cap \{y, r, s, z\} = \{y, z\}$. Same argument as in case v.

Case vii: $S \cap \{y, r, s, z\} = \{y, r, z\}$. Then $\partial S = \partial'(S - r) + a + b$ and so $J' \cap \partial S = J' \cap (\partial'(S - r) + a + b) = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. In particular, $(J' + a) \cap \partial S \neq \emptyset$ and $(J' + b) \cap \partial S \neq \emptyset$. In addition, $(J' + a - c + d) \cap \partial S \supseteq \{d\} \cap \partial S \neq \emptyset$ and $(J' + b - d + c) \cap \partial S \supseteq \{c\} \cap \partial S \neq \emptyset$. □

Proposition 7.13 (joins for pair of alternating open triplets) *Let (y, r, z) and (y, s, z) be alternating open triplets in a digraph D. Suppose that $a := yr$ and $b := zr$ are the legs of the first triplet and $d := sy$ and $c := sz$ are the legs of the second triplet. Let $D' := D - r$. For any join J' of D', both $J' + a$ and $J' + b$ as well as $J' + a - c + d$ and $J' + b - d + c$ are joins of D.*

PROOF: Let J' be a join of D' and S a nontrivial source of D. For each of the six possible values of the intersection $S \cap \{y, r, s, z\}$ we argue next that $J' + a$ and $J' + b$ as well as $(J' + a - c + d) \cap \partial S$ and $(J' + b - d + c) \cap \partial S$ are nonempty.

Case i: $S \cap \{y, r, s, z\} = \emptyset$. Then $\partial S = \partial' S$ and therefore $(J' - c - d) \cap \partial S = J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $J' + a$ and $J' + b$ as well as $(J' + a - c + d) \cap \partial S$ and $(J' + b - d + c) \cap \partial S$ are nonempty.

Case ii: $S \cap \{y, r, s, z\} = \{y, r, s, z\}$. Then $\partial S = \partial'(S - r)$ and therefore $(J' - c - d) \cap \partial S = J' \cap \partial S = J' \cap \partial'(S - r) \neq \emptyset$ since $S - r$ is a nontrivial source of D'. In particular, $J' + a$ and $J' + b$ as well as $(J' + a - c + d) \cap \partial S$ and $(J' + b - d + c) \cap \partial S$ are nonempty.

Case iii: $S \cap \{y, r, s, z\} = \{y, s\}$. Then $\partial S = \partial' S + a$ and therefore $J' \cap \partial S = J' \cap (\partial' S + a) =$

$J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J'+a) \cap \partial S$ and $(J'+b) \cap \partial S$ are nonempty. In addition, $(J'+a-c+d) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$ and $(J'+b-d+c) \cap \partial S \supseteq \{c\} \cap \partial S \neq \emptyset$.

Case iv: $S \cap \{y,r,s,z\} = \{z,s\}$. Then $\partial S = \partial' S + b$ and therefore $J' \cap \partial S = J' \cap (\partial' S + b) = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J'+a) \cap \partial S$ and $(J'+b) \cap \partial S$ are nonempty. In addition, $(J'+a-c+d) \cap \partial S \supseteq \{d\} \cap \partial S \neq \emptyset$ and $(J'+b-d+c) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$.

Case v: $S \cap \{y,r,s,z\} = \{y,s,z\}$. Then $(J'+a) \cap \partial S \supseteq \{a\} \cap \partial S \neq \emptyset$ and $(J'+b) \cap \partial S \supseteq \{b\} \cap \partial S \neq \emptyset$. Similarly, $(J'+a-c+d) \cap \partial S \neq \emptyset$ and $(J'+b-d+c) \cap \partial S \neq \emptyset$.

Case vi: $S \cap \{y,r,s,z\} = \{s\}$. Then $\partial S = \partial' S$ and so $J' \cap \partial S = J' \cap \partial' S \neq \emptyset$ since S is a nontrivial source of D'. In particular, $(J'+a) \cap \partial S \neq \emptyset$ and $(J'+b) \cap \partial S \neq \emptyset$. In addition, $(J'+a-c+d) \cap \partial S \supseteq \{d\} \cap \partial S \neq \emptyset$ and $(J'+b-d+c) \cap \partial S \supseteq \{c\} \cap \partial S \neq \emptyset$. □

A Appendix

This appendix establishes some notational conventions and lists the definitions of standard terms in graph theory.

A.1 Notational conventions

For any set S and any objects s and s', we denote by $S - s$ the set $S \setminus \{s\}$ and by $S - s - s'$ the set $(S - s) - s'$. (If $s \notin S$ then, of course, $S - s = S$.) For any object t, we denote by $S + t$ the set $S \cup \{t\}$. (If $t \in S$ then, of course, $S + t = S$.) Similarly, we denote by $S - s + t$ the set $(S - s) + t$.

A.2 Graph theory terminology

A ***graph***, or ***undirected graph***, is a pair (V, E) of sets where V is a finite set of ***vertices*** and E is a set of unordered pairs of vertices. Each element of E is an ***edge***. An edge of the form $\{v, w\}$ can be denoted by vw or wv. The vertices v and w are the ***ends*** of edge vw. The sets of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$ respectively.

Clearly, the two ends of every edge are distinct and two different edges cannot have the same pair of ends. In other words, graphs have no loops and no parallel edges.

Two vertices v and w of a graph are ***adjacent*** if vw is an edge. If v and w are adjacent, we say that v is a ***neighbor*** of w. An edge is said to be ***incident*** to both its ends. The ***degree*** of a vertex v is the number of edges incident to v.

A graph is ***complete*** if every pair of its vertices is an edge. Any complete graph on n vertices is denoted by K_n.

A ***path*** in a graph is any sequence $(v_0, v_1, v_2, \ldots, v_k)$ of pairwise distinct vertices such that v_i is adjacent to v_{i-1} for $i = 1, 2, \ldots, k$. Vertex v_0 is the ***origin*** and vertex v_k is the ***terminus*** of the path. The vertices v_1, v_2, …, v_{k-1} are ***internal*** to the path. An ***edge of*** the path is any edge of the form $v_{i-1}v_i$. The set of edges of a path P is denoted by $E(P)$. The ***length*** of path P is the number $|E(P)|$.

A path ***from*** v ***to*** w is any path with origin v and terminus w. Two paths are ***internally disjoint*** if they have to internal vertices in common.

A ***circuit*** in a graph is any sequence $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k)$ of vertices such that $k \geq 3$, $v_k = v_0$, the vertices v_0, v_1, …, v_{k-1} are pairwise distinct, and vertex v_i is adjacent to v_{i-1} for $i = 1, 2, \ldots, k$. An

DRAFT

edge of the circuit is any edge of the form $v_{i-1}v_i$. The sets of vertices and edges of a circuit C are denoted by $V(C)$ and $E(C)$ respectively. The ***length*** of circuit C is the number $|E(C)|$.

Given a circuit $(v_0, v_1, \ldots, v_{k-1}, v_k)$, any sequence $(v_i, v_{i+1}, \ldots, v_k, v_1, \ldots, v_{i-1}, v_i)$ is also a circuit. We regard all such sequences as equivalent representations of the circuit.

A graph is ***connected*** if, for any two vertices s and t, there exists a path from s to t. A vertex v of a connected graph G is a ***cut vertex*** if $G - v$ is disconnected.

A graph with more than two vertices is ***2-connected*** if, for any two vertices s and t, there are two internally disjoint paths from s to t. A graph with more than two vertices is 2-connected if and only if it is connected and has no cut vertices.

A ***subgraph*** of a graph G is any graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any subset E' of $E(G)$, we denote by $G - E'$ the subgraph $(V(G), E(G) \setminus E')$. For any subset V' of $V(G)$, we denote by $G - V'$ the subgraph $(V(G) \setminus V', E(G) \setminus E')$, where E' is the set of all edges of G that have at least one end in V'.

For any edge e and any vertex v of a graph G, the subgraphs $G - \{e\}$ and $G - \{v\}$ are denoted by $G - e$ and $G - v$ respectively. For any two vertices v and w of G, the graph $(V(G), E(G) + vw)$ is denoted by $G + vw$.

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DRAFT

Index

DRAFT

DRAFT