Woodall's conjecture on packing of dijoins*

Paulo Feofiloff
https://www.ime.usp.br/~pf/dijoins/

21/10/2005, rewritten 8/1/2016

Abstract

If every directed cut of a graph has k or more edges then the graph has k or more pairwise disjoint dijoins. This is Woodall's conjecture [Sch03]. Here, a *dijoin* is any set of edges that intersects every directed cut; a *directed cut* is the set of the edges that leave some source; and a *source* is a set of vertices that no edge enters. This talk presents the basics of the conjecture, its "capacitated" version, the counterexample of Schrijver, the counterexamples of Cornuéjols and Guenin, and some results by Williams [Wil04].

1 Introduction

A ***directed graph***, or simply ***graph***, is a pair (V, E) where V is a finite set and E is a set of ordered pairs of elements of V. The elements of V are called ***vertices*** and those of E are called ***edges***. For each edge vw, the vertex v is the ***positive end*** of the edge and w is the ***negative end***. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The ***transpose***, or ***directional dual***, of a graph G is the graph obtained by replacing each edge vw by the pair wv.

Cuts. For any set X of vertices, we denote by $\partial^+(X)$ the set of edges that leave X (i.e., have positive end in X and negative end out of X) and by $\partial^-(X)$ the set of edges that enter X. A ***source*** is any X of vertices such that $\partial^-(X) = \emptyset$. A ***sink*** is a source in the transpose of G. The sources $\emptyset$ and $V(G)$ are ***trivial***. A ***source vertex*** is any vertex s such that $\{s\}$ is a source and a ***sink vertex*** is a source vertex in the transpose of G.

A ***directed cut***, or simply ***cut***, is any set of the form $\partial^+(X)$ such that X is a nontrivial source. We say that X is a ***positive shore*** of the cut and $V(G) \setminus X$ is the corresponding ***negative shore***. A graph is ***connected*** if $\emptyset$ is not a cut. In a connected graph, every cut has a unique positive shore and a unique negative shore.

[*] Revised version of a 2005 talk at IME–USP.

Joins. A ***dijoin***, or simply ***join***, is any set of edges that intersects all the cuts, i.e., any subset J of $E(G)$ such that $J \cap C \neq \emptyset$ for every cut C. A graph has a join if and only if $\emptyset$ is not a cut. Moreover, $\emptyset$ is a join if and only if the graph has no cut.

The following characterization is useful: a set J of edges is a join if and only if for every pair (s, t) of vertices there is a path from s to t whose forward edges[1] belong to J. This characterization can also be formulated as follows: a set J of edges is a join if and only if the contraction of all the edge in the set makes the graph strongly connected.[2]

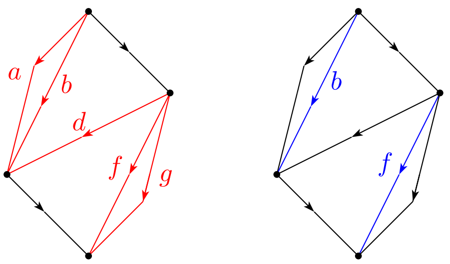


Figure 1: On the left, the set of edges $\{a, b, d, f, g\}$ is a cut. On the right, the set of edges $\{b, f\}$ is a join.

Cuts versus packings of joins. A set $\mathcal{P}$ of joins is ***disjoint*** if the elements of $\mathcal{P}$ are pairwise disjoint. In other words, $\mathcal{P}$ is disjoint if each edge of the graph belongs to at most one element of $\mathcal{P}$. A ***packing of joins*** is a disjoint set of joins. There is no harm in assuming that each join in the packing is minimal.[3]

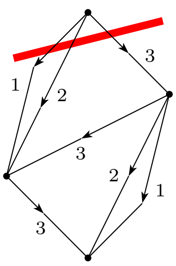


Figure 2: The colored line indicates a cut of size 3. The labels 1, 2 and 3 indicate a packing of three joins.

There is an obvious relation between the size of a cut and the size of a packing of joins:

> **Lemma 1.1** *The inequality $|\mathcal{P}| \leq |C|$ holds for any packing $\mathcal{P}$ of joins and any cut C.* ■

The following conjecture by Woodall [Woo78a, Woo78b, Sch03] remains open:

---

[1] An edge vw of a path is ***forward*** if the path traverses the edge from v to w and ***reverse*** if the path traverses the edge from w to v.

[2] A graph is ***strongly connected*** if for each pair (s, t) of its vertices there exists a path from s to t without forward edges.

[3] A join is ***minimal*** if none of its proper subsets is a join.

Conjecture 1 (Woodall) *Every graph has a packing $\mathcal{P}$ of joins and a cut C such that $|\mathcal{P}| = |C|$.*

This conjecture is dual to the Lucchesi–Younger theorem [LY78], according to which every connected graph has a packing $\mathcal{C}$ of cuts and a join J such that $|\mathcal{C}| = |J|$.

Dags. Every edge of a graph belongs to a cut or to a directed circuit,[4] but not both. In particular, $C \cap E(Z) = \emptyset$ for every cut C and every directed circuit Z, where $E(Z)$ is the set of edges of Z. As a consequence, we can restrict our attention to the joins that do not use edges of directed circuits. Hence, conjecture 1 can be restricted to graphs that do not have directed circuits, i.e., to ***dags*** (*directed acyclic graphs*).

2 Minimum cuts and maximum packings of joins

A cut C is ***minimum*** if there is no cut C' such that $|C'| < |C|$. A packing $\mathcal{P}$ of joins is ***maximum*** if there is no packing $\mathcal{P}'$ such that $|\mathcal{P}'| > |\mathcal{P}|$. Woodall's conjecture invites the consideration of the following pair of optimization problems:

Problem 2.1 *Find a minimum cut in a graph.*

Problem 2.2 *Find a maximum packing of joins in a graph.*

There is a polynomial algorithm for problem 2.1 (it is a variant of the Max-flow Min-cut algorithm). No polynomial algorithm is known for problem 2.2, but there is no evidence to suggest that the problem is NP-hard.

It is convenient to have a notation for the size of the objects that the two problems deal with. Given a graph G, let

$$\nu(G)$$

denote the size of a maximum packing of joins of G and let

$$\tau(G)$$

denote the size of a minimum cut of G. If G is disconnected then $\tau(G) = 0$ (since $\emptyset$ is a cut) and $\nu(G) = 0$ (since there are no joins). If G has a cut then $\tau(G)$ and $\nu(G)$ are finite. If G has no cut then $\tau(G) = \infty$ and $\nu(G) = \infty$ (since any number of copies of $\emptyset$ is a packing of joins).

An immediately consequence of lemma 1.1 is that $\nu(G) \leq \tau(G)$ for every graph G. Conjecture 1 can then be formulated as follows:

Conjecture 2 (Woodall) *For every graph G one has $\nu(G) = \tau(G)$.*

We say that a graph G ***satisfies Woodall's conjecture*** if $\nu(G) = \tau(G)$. It is obvious that every graph G with $\tau(G) \leq 1$ satisfies Woodall's conjecture. It is less obvious that every graph

[4] A circuit is ***directed*** if it has no reverse edges.

G with $\tau(G) = 2$ satisfies the conjecture [Sch03, p.968]. It is also known [FY87, Sch82] that every dag with a single source vertex (or a single sink vertex) satisfies the conjecture.

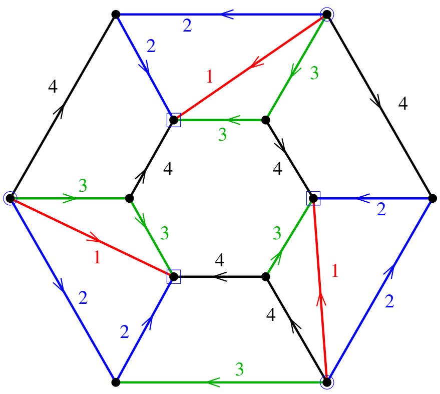


Figure 3: This graph has $\nu = 4$ and $\tau = 4$ and therefore satisfies Woodall's conjecture. The colors (and the numerical labels) indicate a packing of 4 joins. The graph is a dag. The source vertices are indicated by circles and the sink vertices by squares.

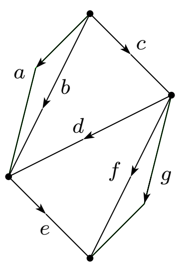
3 Linear programs

Let $\mathcal{J}$ be the set of all the joins of a graph $G = (V, E)$ and M the matrix indexed by $\mathcal{J} \times E$ whose rows are the characteristic vectors of the elements of $\mathcal{J}$. Now consider the following dual pair of linear programs:

$$\text{maximize} \quad y1 \quad \text{subject to } y \in \mathbb{R}^{\mathcal{J}}_{+} \text{ and } yM \leq 1\,, \tag{1}$$

$$\text{minimize} \quad 1x \quad \text{subject to } x \in \mathbb{R}^{E}_{+} \text{ and } Mx \geq 1\,. \tag{2}$$

(The "1" represents a vector whose elements are all equal to 1. The vector is indexed by $\mathcal{J}$ or by E, depending on the context.)



a	b	c	d	e	f	g
1	–	–	–	–	1	–
1	–	–	–	–	–	1
–	1	–	–	–	1	–
–	1	–	–	–	–	1
–	–	1	1	1	–	–
1	–	–	–	1	–	–
–	1	–	–	1	–	–
–	–	1	–	–	1	–
–	–	1	–	–	–	1

a	b	c	d	e	f	g
1	1	1	–	–	–	–
1	1	–	1	–	1	1
–	–	–	–	1	1	1

Figure 4: The rows of the first matrix are the characteristic vectors of all the minimal joins of the graph. The graph is a dag and has only one source and one sink. The rows of the second matrix are the characteristic vectors of all the minimal cuts.

If "$y \in \{0,1\}^{\mathcal{J}}$" is substituted for "$y \in \mathbb{R}^{\mathcal{J}}_{+}$" in linear program (1), we have an integer program that represents problem 2.2. Every feasible vector y in this program represents a packing of joins and $y1$ is the size of the packing. The optimum value of this program is $\nu(G)$.

If “$x \in \{0,1\}^E$” is substituted for “$x \in \mathbb{R}^E_+$” in linear program (2), we have an integer program that represents problem 2.1. Every x in this program is the characteristic vector of a cut (since a cut is, precisely, a set of edges that intersects all the joins) and $1x$ is the size of the cut. The optimum value of this program is $\tau(G)$.

It follows from the Lucchesi–Younger theorem [LY78] that all the vertices of the polyhedron $\{x : x \in \mathbb{R}^E_+ \text{ and } Mx \geq 1\}$ are integer and therefore every solution of the linear program (2) belongs to $\{0,1\}^E$. Hence, $\tau(G) = \nu^*(G)$, where $\nu^*(G)$ is the optimum value of linear program (1).

4 Analogy with maximum flow

Woodall's conjecture is similar to the Max-flow Min-cut theorem [Sch03, cap.10]. That theorem holds for any graph and any pair (s,t) of its vertices, and states that the size of a maximum flow from s to t is equal to the size of a minimum semi-cut separating s from t. Here, a ***flow*** is a set of directed[5] paths from s to t with no edges in common. And a ***semi-cut*** is any set of the form $\partial^+(X)$ such that $s \in X \subseteq V(G) \setminus \{t\}$ (the set X is not required to be a source).

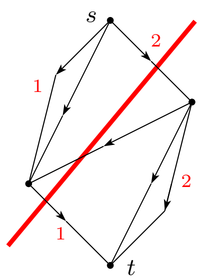


Figure 5: A maximum flow (labels 1 and 2) and a minimum semi-cut (colored line).

The similarity between Woodall's conjecture and the Max-flow Min-cut theorem is not perfect. In the theorem, the paths are directed and there are two fixed vertices. In the conjecture, there are no fixed vertices, the paths (that represent joins) are not necessarily directed, and only the forward edges of the paths are taken into account.

The Max-flow Min-cut theorem has a generalization in which each edge a has a capacity c_a in $\mathbb{N}$, where $\mathbb{N}$ is the set $\{0,1,2,3,\ldots\}$ of natural numbers. An edge a cannot be used more than c_a times by the flow and contributes c_a to the size of each semi-cut that contains a. It is hard to imagine that the Max-flow Min-cut theorem could be true without its capacitated generalization being also true.

The similarity between the Max-flow Min-cut theorem and Woodall's conjecture suggests that one should study the capacitated generalization of the conjecture.

5 Capacitated generalization of Woodall's conjecture

A ***capacitated graph*** is a pair (G,c) where G is a graph and c is a vector indexed by $E(G)$ with values in $\mathbb{N} \cup \{\infty\}$. This vector attributes a ***capacity*** c_a to each edge a of G. The edge

[5] A path is ***directed*** if it has no reverse edges.

a is ***null*** if $c_a = 0$ and ***infinite*** if $c_a = \infty$. Attributing infinite capacity to an edge has the same effect as contracting the edge to a single vertex.

The presence of infinite edges requires a redefinition of the terms "source", "cut" and "join". A ***source of*** (G, c) is a source X of G such that $\partial^+(X)$ has no infinite edges. A ***cut of*** (G, c) is any set of the form $\partial^+(X)$ such that X is a nontrivial source of (G, c). In other words, a cut of (G, c) is any cut of G having no infinite edges. A ***join of*** (G, c) is a set of edges that intersects all the cuts of (G, c) and has no infinite edges.

The ***capacity*** of a cut C of (G, c) is the number $c(C) := \sum_{a \in C} c_a$. A cut C is ***minimum*** if there is no cut C' of (G, c) such that $c(C') < c(C)$.

Since the capacity of an edge can be greater than 1, it is natural to consider *multisets* of joins, i.e., sets that may contain several copies of one and the same join, with each copy contributing 1 to the size of the multiset. A multiset $\mathcal{P}$ of joins is ***disjoint in*** (G, c) if

$$|\mathcal{P}(a)| \leq c_a$$

for each edge a, where $\mathcal{P}(a)$ is the multiset $\{J \in \mathcal{P} : J \ni a\}$. In other words, $\mathcal{P}$ is disjoint if each edge a belongs to at most c_a elements of $\mathcal{P}$. (In particular, if $c_a = 0$ then no element of $\mathcal{P}$ contains a.)

A ***packing*** of joins ***in*** (G, c) is a disjoint multiset of joins in (G, c). The following relation between cuts and packings of joins generalizes lemma 1.1:

Lemma 5.1 *In any capacitated graph (G, c), for any packing $\mathcal{P}$ of joins and any cut C,*

$$|\mathcal{P}| \leq c(C).$$

If $|\mathcal{P}| = c(C)$ then $|J \cap C| = 1$ for each J in $\mathcal{P}$ and $|\mathcal{P}(a)| = c_a$ for each a in C.

PROOF: Let $\mathcal{P}$ a packing of joins and C be a cut of (G, c). For each element J of $\mathcal{P}$ there is an edge a of C such that $\mathcal{P}(a) \ni J$. Hence,

$$|\mathcal{P}| \;\leq\; \sum_{a \in C} |\mathcal{P}(a)| \;\leq\; \sum_{a \in C} c_a \;=\; c(C).$$

Suppose now that $|\mathcal{P}| = c(C)$. Then the first "$\leq$" holds as "$=$" and therefore $|J \cap C| = 1$ for each J in $\mathcal{P}$. The second "$\leq$" also holds as "$=$", whence $|\mathcal{P}(a)| = c_a$ for each a in C.

The definition of parameters τ and ν must be adjusted to take into account the capacities of the edges. We denote by

$$\nu(G, c) \quad \text{and} \quad \tau(G, c)$$

the size of a maximum packing of joins of (G, c) and the capacity of a minimum cut of (G, c) respectively. Lemma 5.1 has the following immediate consequence: every capacitated graph (G, c) satisfies the inequality

$$\nu(G, c) \leq \tau(G, c). \tag{3}$$

The corresponding generalization of Woodall's conjecture (conjecture 2) is known as Edmonds–Giles's conjecture [EG77]:

Conjecture 3 (Edmonds–Giles) *Every capacitated graph (G, c) satisfies the equality $\nu(G, c) = \tau(G, c)$.*

If $\tau(G,c) = 0$ then $\nu(G,c) = 0$ and therefore $\nu(G,c) = \tau(G,c)$. If $\tau(G,c) = 1$ then $\{a \in E(G) : 0 < c_a < \infty\}$ is a join, hence $\nu(G,c) \geq 1$, and so $\nu(G,c) = \tau(G,c)$ by virtue of (3). Therefore, the conjecture is true when restricted to the instances where $\tau(G,c) \leq 1$.

Null edges. The capacitated generalization of the Max-flow Min-cut theorem (see section 4) can be reduced to the original, uncapacitated, version. The reduction consists of deleting the edges of capacity 0 and replacing each edge of capacity $k \geq 2$ with k edges in parallel. At first sight, it would seem that the same construction could reduce Edmonds–Giles's conjecture to Woodall's conjecture. It is true that an edge a of capacity $k \geq 2$ can to be emulated by k copies of a in parallel. But the deletion of an edge can create new cuts[6] and thereby change the instance of the problem. Hence, Edmonds–Giles's conjecture does not reduce to Woodall's conjecture.

6 Counterexamples

The conjecture of Edmonds–Giles is false. A ***counterexample*** to the conjecture is any capacitated graph (G,c) such that $\nu(G,c) < \tau(G,c)$. The next sections will show several counterexamples. They all have null edges and therefore do not affect conjecture 2.

We shall say that a capacitated graph (G,c) ***is good*** if there is no c such that (G,c) is a counterexample. Conjecture 3 could be formulated by saying "every graph is good". It is known, for example, that

1. every dag with a single source vertex is good and
2. every source-sink connected[7] dag is good.

The proof of 1 is analogous to the proof of the Max-flow Min-cut theorem mentioned in section 4. This proof leads to a polynomial algorithm to compute $\tau(G,c)$. The proof of 2 was found by Schrijver [Sch82] and, independently, by Feofiloff and Younger [FY87].

7 The counterexample of Schrijver

Schrijver [Sch80] found the first counterexample to conjecture 3. The counterexample is represented in figure 6 and will be denoted by (G_1, c_1).

Fact 7.1 $\nu(G_1,c_1) = 1$ *and* $\tau(G_1,c_1) = 2$.

PROOF: It is easy to check that $\tau(G_1,c_1) = 2$ and that one of the two shores of each minimum cut has only one vertex.

Let A_1 be the set of ***active*** edges, i.e., edges whose capacity is 1. Notice that the subgraph induced by A_1 consists of three paths, each of length 3. These are the ***active paths*** of the graph. We shall say that a cut is ***critical*** if it intersects each active path only once. Figure 7 shows that there are four critical cuts.

[6] The deletion of an edge creates new cuts if and only if the edge is not transitive. An edge vw is ***transitive*** if there exists a directed path from v to w in $G - vw$.

[7] A dag is ***source sink connected*** if each source vertex is connected to each sink vertex by a directed path.

Suppose, for a moment, that $\nu(G_1, c_1) \geq 2$. Then A_1 contains two mutually disjoint joins, say J and K. The edges of each active path alternate between J and K, since each internal vertex of each active path is the shore of a cut with exactly 2 active edges. In other words, each active path follows the pattern (J, K, J) or the pattern (K, J, K). In the aggregate of the three active paths, these two patterns can be combined in only 4 different ways, as indicated in figure 7. But, for each of the 4 combinations, either J or K does not intersect one of the critical cuts. Hence, J or K is not a join, contrary to what we were assuming. This contradiction shows that $\nu(G_1, c_1) < 2$. Since A_1 is a join, $\nu(G_1, c_1) = 1$.

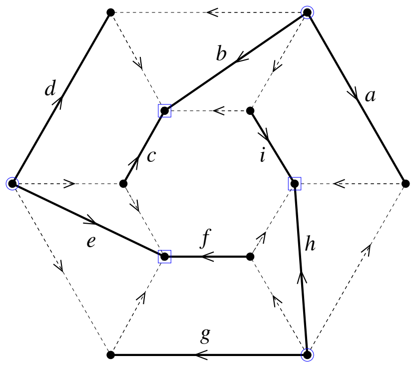


Figure 6: Schrijver's counterexample (G_1, c_1) to conjecture 3. The capacity vector c_1 has values in $\{0, 1\}$. The zero capacity edges are indicated by dashed lines; the remaining edges are indicated by solid lines. The graph is a dag; the source vertices are indicated by circles and the sink vertices by squares.

a	b	c	d	e	f	g	h	i
J	K	J	J	K	J	J	K	J
J	K	J	J	K	J	K	J	K
J	K	J	K	J	K	J	K	J
J	K	J	K	J	K	K	J	K

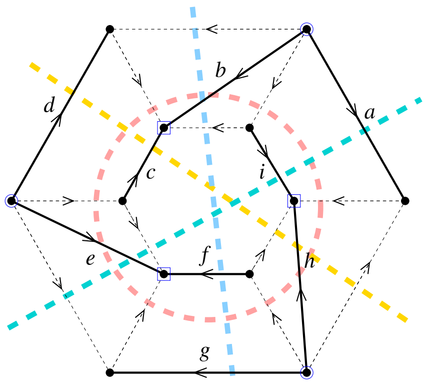


Figure 7: Each row of the table shows a possible arrangement of two potential mutually disjoint joins, J and K, in the capacitated graph (G_1, c_1) of figure 6. In each row of the table, one of J and K does not intersects one of the four critical cuts indicated in the drawing.

Schrijver's capacitated graph has the shape of a ring of length $2i$, with $i = 3$. The analogous capacitated graphs with $i = 5, 7, 9, \ldots$ (see figure 8) are also counterexamples. But the analogous capacitated graphs with $i = 2, 4, 6, 8, \ldots$ are not counterexamples.

7.1 Fractional packing of joins

The capacitated graph (G_1, c_1) in figure 6 has no packing of size 2. Curiously, (G_1, c_1) has a "fractional packing" of size 2, as we show next.

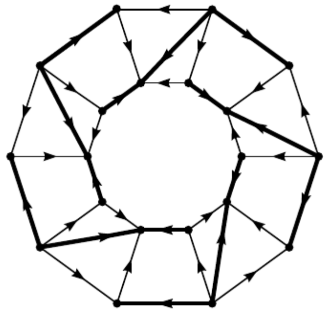


Figure 8: Generalization of (G_1, c_1) based on a ring of length 2×5. (Figure copied from Williams [Wil04].) This graph is a counterexample.

Let's say that the four joins $\{a, c, d, f, h\}$, $\{d, f, g, i, b\}$, $\{g, i, a, c, e\}$ and $\{b, h, e\}$ are ***special***. Give weight $\frac{1}{2}$ to each special join and weight 0 to all the other joins of G_1. Each edge of capacity 1 in (G_1, c_1) belongs to exactly two of the special joins and each edge of capacity 0 belongs to no special join. Thus, the sum of the weights of all the joins that contain a given edge a is no greater that the capacity of a. One can say then that the weighted set of special joins is disjoint in (G_1, c_1). The size of this weighted set is the sum of the weights of all the joins, that is, $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$. Summarizing: (G_1, c_1) has a "fractional packing" of size 2.

This example illustrates a general phenomenon. For any capacitated graph (G, c), consider the dual pair of linear programs

$$\text{maximize} \quad y1 \quad \text{subject to } y \in \mathbb{R}_+^{\mathcal{J}} \text{ and } yM \le c \tag{4}$$

$$\text{minimize} \quad cx \quad \text{subject to } x \in \mathbb{R}_+^{E} \text{ and } Mx \ge 1 \tag{5}$$

that generalize the programs (2) and (1) in section 3. It can be shown that $\nu^*(G, c) = \tau(G, c)$, where $\nu^*(G, c)$ is the optimal value of program (4) and $\tau(G, c)$ is the optimal value of program (5).

8 The counterexamples of Cornuéjols and Guenin

For two decades, (G_1, c_1) was the only known counterexample for conjecture 3. In 2002, Cornuéjols and his student Guenin [CG02] found two new counterexample, that we shall denote by (G_2, c_2) and (G_3, c_3). These counterexamples are represented in figures 9 and 10 respectively.

Fact 8.1 $\nu(G_2, c_2) = 1$ *and* $\tau(G_2, c_2) = 2$. ■

Fact 8.2 $\nu(G_3, c_3) = 1$ *and* $\tau(G_3, c_3) = 2$. ■

The proofs of these facts are similar to the proof of fact 7.1. Figure 11 shows the critical cuts used in the proofs. (These are the cuts that intersect each active path only once.)

Some simple variations of (G_2, c_2) and (G_3, c_3) are also counterexamples. Williams [Wil04] discusses several variations. For example, if u and x are the vertices 14 and 8 in figure 9 then $(G_2 - ux, c_2')$ is a counterexample, where c_2' is the restriction of c_2 to the set of edges of $G_2 - ux$.

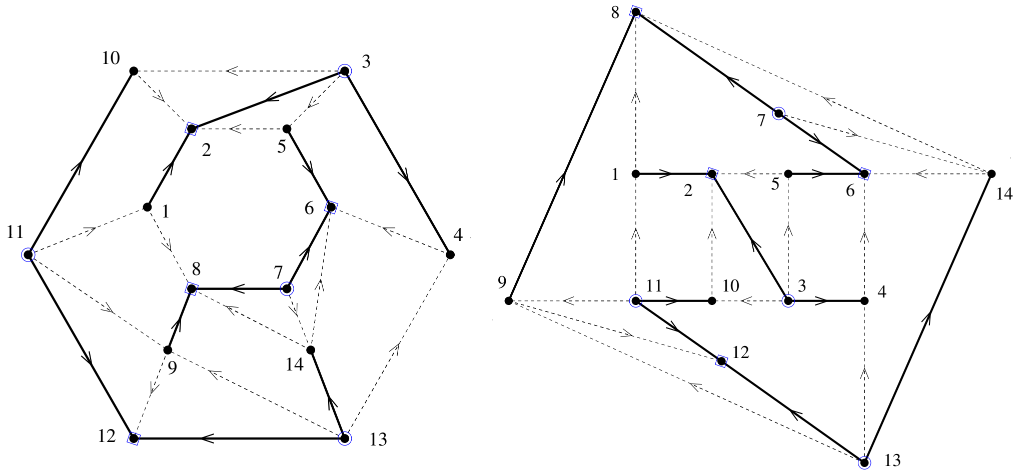


Figure 9: Two drawings of the counterexample (G_2, c_2) by Cornuéjols and Guenin. Vector c_2 has values in $\{0, 1\}$. The edges of null capacity are indicated by dashed lines; the remaining ones, by solid lines. The graph is a dag.

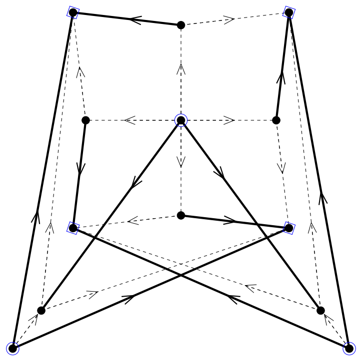


Figure 10: The counterexample (G_3, c_3) of Cornuéjols and Guenin. The vector c_3 has values in $\{0, 1\}$. The edges of null capacity are indicated by dashed lines; the remaining ones, by solid lines. The graph is a dag.

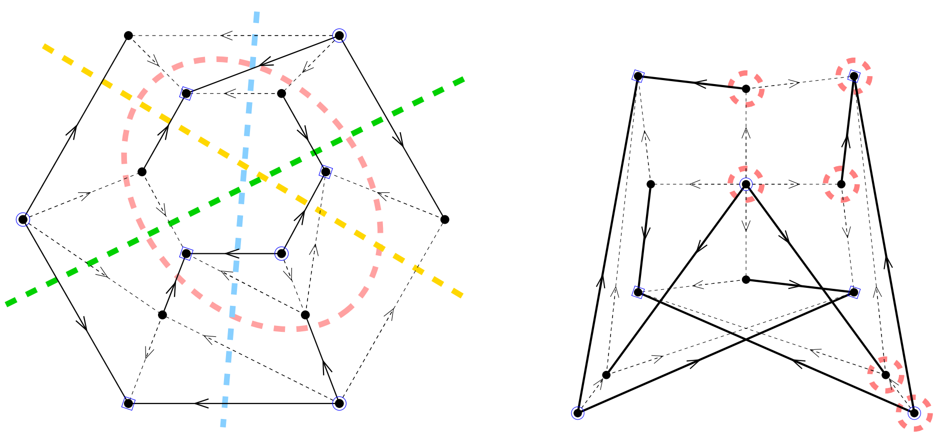


Figure 11: The first drawing indicates the four critical cuts of (G_2, c_2). The second drawing indicates the vertices of the positive shore of one of the critical cuts of (G_3, c_3); the other three critical cuts are defined by symmetry.

9 Minimal counterexamples

To make a catalog of counterexamples of Edmonds–Giles's conjecture (conjecture 3), we can limit ourselves to the counterexamples that, in some sense, do not "contain" other counterexamples. We say that such counterexamples are "minimal".

To specify the conditions for a counterexample to "contain" another, we must introduce an order relation between capacity vectors. Given capacity vectors c and c' for a graph G, we say that $c' < c$ if $c'_a \leq c_a$ for every edge a but $c'_a < c_a$ for some edge a. The relation $<$ is, of course, transitive (i.e., if $c'' < c'$ and $c' < c$ then $c'' < c$) and anti-symmetric (i.e., if $c' < c$ then $c \not< c'$).

We must also introduce some auxiliary notation. For any capacitated graph (G, c), denote by $I(G, c)$ and $N(G, c)$ respectively the sets of all infinite and all null edges.

We can now define an inclusion relation between capacitated graphs. We say that a capacitated graph (G', c') ***is contained in*** (G, c) if

i. $V' = V$ and $E' \subset E$ or
ii. $V' = V$ and $E' = E$ and $I' \supset I$ or
iii. $V' = V$ and $E' = E$ e $I' = I$ and $N' \supset N$ or
iv. $V' = V$ and $E' = E$ and $I' = I$ and $N' = N$ and $c' < c$,

where V', E', I' and N' are abbreviations for $V(G')$, $E(G')$, $I(G', c')$ and $N(G', c')$ respectively and where the abbreviations V, E, I and N are defined likewise for (G, c). The is-contained-in relation between capacitated graphs is transitive and anti-symmetric.

We can finally define the concept of minimal counterexample. We say that a counterexample (G, c) is ***minimal*** if no counterexample (G', c') is contained in (G, c).

If $E(G) = \emptyset$ or $I(G, c) \cup N(G, c) = E(G)$ then (G, c) is not a counterexample. From this and from the transitivity and anti-symmetry of the is-contained-in relation, every non minimal counterexample contains a minimal counterexample.

We show next some examples to illustrate the concept:

1. The counterexample (G_2, c_2) of Cornuéjols–Guenin (figure 9) is not minimal, as Williams [Wil04] observed. Indeed, if we denote by u e x the vertices 14 and 8 in figure 9 and denote by c'_2 the restriction of c_2 to the set of edges of $G_2 - ux$ then $(G_2 - ux, c'_2)$ is a counterexample, since $\nu(G_2 - ux, c'_2) = \nu(G_2, c_2) < \tau(G_2, c_2) = \tau(G_2 - ux, c'_2)$. Moreover, $(G_2 - ux, c'_2)$ is contained in (G_2, c_2).
2. The counterexample (G'_1, c'_1) in figure 8 contains the capacitated graph (G''_1, c''_1) in figure 12. The latter is a counterexample because it is "equivalent" to (G_1, c_1) in figure 6. Therefore, the counterexample (G'_1, c'_1) is not minimal.
3. The counterexample (G_1, c_1) of Schrijver (see figure 6) is minimal, even if this is not obvious.

10 Some properties of minimal counterexamples

Williams [Wil04, WG05] has shown that every minimal counterexample (G, c) has the following properties:

1. no null edge is transitive,

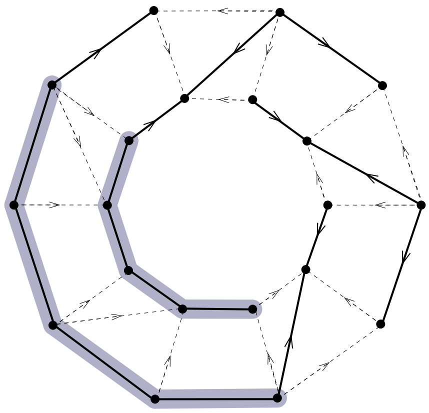


Figure 12: Capacitated graph (G_1'', c_1''). (Compare with (G_1', c_1') in figure 8.) The gray bands indicate edges of infinite capacity. The orientation of these edges was omitted since they can be traversed in any direction.

2. every active edge belongs to a minimum cut,
3. the edges of every directed circuit are infinite,
4. every minimum cut is peripheral,
5. every circuit has some null or infinite edge.

The technical terms in this list of properties have the following definitions: An edge vw is ***transitive*** in (G, c) if there is a directed path from v to w in $(G - vw, c')$, where c' is the restriction of c to the set of edges of $G - vw$. An edge a is ***active*** in (G, c) if it is neither null nor infinite. A cut C is ***peripheral*** in (G, c) if, for one of the two shores of C, all the edges that have both ends in that shore are infinite. (For example, C is peripheral if the positive shore or the negative shore of C has only one vertex.)

The counterexamples (G_1, c_1), (G_2, c_2) e (G_3, c_3) in figures 6, 9 and 10 have the properties 1 to 5, even though not all of them are minimal. The counterexample (G_1'', c_1'') in figure 12 has several null transitive edges and is therefore not minimal.

Next, we give the proof of each property listed above.

10.1 Elimination of null transitive edges

The deletion of null transitive edges does not create new cuts and does not change the values of the parameters ν and τ.

Proposition 10.1 *Minimal counterexamples do not have null transitive edges.*

PROOF: Let (G, c) be a capacitated graph and b a transitive null edge. Let G' be the graph $G - b$ and c' the restriction of c to the set of edges of $G - b$. Let v be the positive end and w the negative end of b. Let B be the set of edges of a directed path from v to w in (G', c'). (Keep in mind that, by definition, all the reverse edges of the path are infinite.)

Note that a cut of G intersects B if and only if it contains b. Moreover, every cut contains at most one edge of B. Hence, G and G' have the same set of sources and therefore (G, c) and (G', c') also have the same set of sources. Moreover, for each source F, we have $c'(C') =$

$c(C)$, where C' e C are the cuts associated with F in G' and G respectively. So,

$$\tau(G', c') = \tau(G, c). \tag{6}$$

Now let $\mathcal{P}$ be a packing of joins of (G, c) and $\mathcal{P}'$ a packing of joins of (G', c'). Since (G, c) and (G', c') have the same set of sources, every join of (G', c') is also a join of (G, c), and therefore $\mathcal{P}'$ is a packing of joins of (G, c). On the other hand, every join of (G, c) that does not contain b is a join of (G', c'). Since $c_b = 0$, no join in $\mathcal{P}$ contains b, and therefore $\mathcal{P}$ is a packing of joins of (G', c'). Hence,

$$\nu(G', c') = \nu(G, c). \tag{7}$$

By virtue of (6) and (7), if (G, c) is a counterexample then (G', c') is also a counterexample. Since $V(G') = V(G)$ and $E(G') \subset E(G)$, the counterexample (G, c) is not minimal.

10.2 Active edges versus minimum cuts

We can reduce the capacity of an active edge that does not belong to a minimum cut. And this operation does not invalidate a counterexample.

Proposition 10.2 *In a minimal counterexample, every active edge belongs to a minimum cut.*

PROOF: Let (G, c) be a counterexample and a a active edge that belongs to no minimum cut. Let c' be the capacity vector defined by

$$c'_a := c_a - 1 \quad \text{and} \quad c'_e := c_e \text{ for each } e \neq a.$$

Of course $I(G, c') = I(G, c)$ and therefore (G, c') and (G, c) have the same set of cuts. Of course $c'(C) = c(C) - 1$ for every cut C that contains a and $c'(C) = c(C)$ for all the remaining cuts. Since no minimum cut of (G, c) contains a, we have

$$\tau(G, c') = \tau(G, c).$$

Now consider the joins. Let $\mathcal{P}'$ be a maximum packing of joins of (G, c'). Since $\mathcal{P}'$ is also a packing in (G, c), we have

$$\nu(G, c') = |\mathcal{P}'| \leq \nu(G, c).$$

But (G, c) is a counterexample, and therefore $\nu(G, c') \leq \nu(G, c) < \tau(G, c) = \tau(G, c')$, whence (G, c') is also a counterexample. Since $N(G', c') \supseteq N(G, c)$ and $c' < c$, the counterexample (G, c) is not minimal.

10.3 Elimination of directed circuits

Contraction of the edges of directed circuits does no change the set of cuts. Hence, we can attribute ∞ to the capacities of these edges.

Proposition 10.3 *In any minimal counterexample, the edges of every directed circuit are infinite.*

PROOF: Let (G, c) be a capacitated graph and O a directed circuit in (G, c). (Reminder: by definition, all the reverse edges of O are infinite.) Suppose $c_a < \infty$ for some forward edge a of O. Define a new capacity vector c' as follows:

$$c'_a := \infty \quad \text{and} \quad c'_e := c_e \text{ for each } e \neq a.$$

Since O is directed in (G, c), no cut of (G, c) contains an edge of O. Hence the set of cuts of (G, c') is identical to the set of cuts of (G, c). Therefore

$$\tau(G, c') = \tau(G, c).$$

No minimal join of (G, c) contains a, since no cut of (G, c) contains a. Hence (G, c') and (G, c) have the same minimal joins. Therefore, every packing of minimal joins in (G, c) is also a packing in (G, c'), and conversely. This shows that

$$\nu(G, c') = \nu(G, c).$$

Suppose now that (G, c) is a counterexample. Then $\nu(G, c) < \tau(G, c)$, hence $\nu(G, c') < \tau(G, c')$, and therefore (G, c') is a counterexample. But $I(c') \supset I(c)$, and therefore the counterexample (G, c) is not minimal.

This proposition shows that every minimal counterexample is essentially a dag.

10.4 Elimination of nonperipheral minimum cuts

Any capacitated graph can be divided into two "independent" capacitated graphs along a minimum nonperipheral cut.

Proposition 10.4 *In any minimal counterexample, every minimum cut is peripheral.*

PROOF: Let C be a minimum cut of a capacitated graph (G, c). Let c' be a capacity vector defined as follows:

$$c'_a := \begin{cases} \infty & \text{if } a \text{ has both ends in the negative shore of } C, \\ c_a & \text{otherwise.} \end{cases}$$

(Informally, c' describes the contraction of the negative shore of C to a vertex.) Define the capacity vector c'' similarly:

$$c''_a := \begin{cases} \infty & \text{if } a \text{ has both ends in the positive shore of } C, \\ c_a & \text{otherwise.} \end{cases}$$

According to lemma 10.1 below, if (G, c) is a counterexample then (G, c') or (G, c'') is a counterexample. On the other hand, if C is non peripheral then $I(c') \supset I(c)$ (since some non infinite edge has both ends in the negative shore of C) and, similarly, $I(c'') \supset I(c)$. Hence, if the counterexample (G, c) is minimal, the cut C must be peripheral.

To finish the proof of the proposition we must establish the following lemma:

Lemma 10.1 *Let C be a minimum cut of a capacitated graph (G, c) and let c' and c'' be the capacity vectors defined at the beginning of the proof of proposition 10.4. If (G, c') and (G, c'') are not counterexamples then (G, c) is not a counterexample.*

PROOF: On one hand, C is a cut of (G, c') (since C has no infinite edges) and $c'(C) = c(C)$, whence $\tau(G, c') \leq c'(C) = c(C) = \tau(G, c)$. On the other hand, $\tau(G, c') \geq \tau(G, c)$ since the set of cuts of (G, c') is part of the set of cuts of (G, c) and the capacity of a cut in (G, c') is equal to capacity of that cut in (G, c). Hence

$$\tau(G, c') = \tau(G, c)$$

and therefore C is a minimum cut of (G, c'). An analogous reasoning shows that $\tau(G, c'') = \tau(G, c)$ and C is a minimum cut of (G, c'').

1. Suppose that (G, c') is not a counterexample, i.e., that $\nu(G, c') = \tau(G, c')$. Let $\mathcal{P}'$ be a maximum packing of joins of (G, c'). Of course $|\mathcal{P}'| = \nu(G, c') = \tau(G, c')$. Since $\tau(G, c') = c'(C)$, we have $|\mathcal{P}'| = c'(C)$. Now lemma 5.1 (see section 5) implies that

$$|\mathcal{P}'(a)| = c_a \quad \text{for each } a \text{ in } C \text{ and} \tag{8}$$
$$|J' \cap C| = 1 \quad \text{for each } J' \text{ in } \mathcal{P}'. \tag{9}$$

Suppose next that (G, c'') is not a counterexample and let $\mathcal{P}''$ be a maximum packing of joins of (G, c''). A reasoning analogous to that of the previous paragraph shows that

$$|\mathcal{P}''(a)| = c_a \quad \text{for each } a \text{ in } C \text{ and} \tag{10}$$
$$|J'' \cap C| = 1 \quad \text{for each } J'' \text{ in } \mathcal{P}''. \tag{11}$$

2. By virtue of (8) and (9), for each nonnull edge a of C, there are elements $J'_{a,1}, \ldots, J'_{a,c_a}$ of $\mathcal{P}'$ such that

$$J'_{a,i} \cap C = \{a\} \tag{12}$$

for $i = 1, \ldots, c_a$. By virtue of (10) and (11), there are elements $J''_{a,1}, \ldots, J''_{a,c_a}$ of $\mathcal{P}''$ such that $J''_{a,i} \cap C = \{a\}$ for $i = 1, \ldots, c_a$. Let

$$J_{a,i} := J'_{a,i} \cup J''_{a,i} \tag{13}$$

for each a in C and each i in $\{1, \ldots, c_a\}$. Given any pair (a, i), let J', J'' and J be abbreviations of $J'_{a,i}$, $J''_{a,i}$ and $J_{a,i}$ respectively. Our next task is to show that J is a join of (G, c), i.e., that $J \cap B \neq \emptyset$ for every cut B of (G, c).

3. Let B be a cut of (G, c) and X the positive shore of B. Let Y be the positive shore of C. If $X \cap Y = \emptyset$ or $X \supseteq Y$ then B is a cut of (G, c''), hence $J'' \cap B \neq \emptyset$, and therefore $J \cap B \neq \emptyset$. If $X \cup Y = V$ or $X \subseteq Y$ then B is a cut of (G, c'), hence $J' \cap B \neq \emptyset$, and therefore $J \cap B \neq \emptyset$. In the remaining cases, due to (9), (11), (12) and (13), lemma 10.2 below shows that $J \cap B \neq \emptyset$. So, J is a join of (G, c).

4. Let $\mathcal{P}$ be the multiset of all the joins $J_{a,i}$ such that a is a nonnull edge of C and i belongs to $\{1, \ldots, c_a\}$. For each edge e of G, if the positive end of e is in the positive shore of C then

$$|\mathcal{P}(e)| \leq c_e$$

since $\mathcal{P}'$ is a packing in (G, c') and $c'_e = c_e$. Similarly, if the negative end of e is in the negative shore of C then $|\mathcal{P}(e)| \leq c_e$. Hence, $\mathcal{P}$ is a packing in (G, c).

5. It follows from the previous paragraph that $\nu(G,c) \geq |\mathcal{P}|$. But $|\mathcal{P}| = |\mathcal{P}'| = |\mathcal{P}''| = \tau(G,c)$, and therefore $\nu(G,c) \geq \tau(G,c)$. Hence, (G,c) is not a counterexample.

To finish the proof of the lemma, we must establish the following consequence of the submodularity of ∂^+:

Lemma 10.2 (submodularity) *Let $\partial^+(Y)$ be a cut of a capacitated graph (G,c). Let J be a set of edges that intersects all the cuts $\partial^+(X)$ of (G,c) for which*

$$X \cup Y = V \quad \text{or} \quad X \cap Y = \emptyset \quad \text{or} \quad X \supseteq Y \quad \text{or} \quad X \subseteq Y.$$

If $|J \cap \partial^+(Y)| = 1$ then J is a join of (G,c).

PROOF: Let X be a nontrivial source of (G,c) such that $X \cup Y \neq V$ and $X \cap Y \neq \emptyset$. To prove that J is a join of (G,c), it is enough to show that $J \cap \partial^+(X) \neq \emptyset$.

Clearly $X \cup Y$ and $X \cap Y$ are nontrivial sources of G. Hence, $\partial^+(X \cup Y)$ and $\partial^+(X \cap Y)$ are cuts of G. These cuts do not have infinite edges, and are therefore cuts of (G,c).

Notice now that the union of $\partial^+(X \cup Y)$ with $\partial^+(X \cap Y)$ is equal to the union of $\partial^+(X)$ with $\partial^+(Y)$ and that the intersection of $\partial^+(X \cup Y)$ with $\partial^+(X \cap Y)$ is equal to the intersection of $\partial^+(X)$ with $\partial^+(Y)$. Hence, the sum $|\partial^+(X \cup Y)| + |\partial^+(X \cap Y)|$ is equal to the sum $|\partial^+(X)| + |\partial^+(Y)|$. It follows then that

$$|J \cap \partial^+(X \cup Y)| + |J \cap \partial^+(X \cap Y)| = |J \cap \partial^+(X)| + |J \cap \partial^+(Y)|. \tag{14}$$

Since $X \cup Y \supseteq Y$ and $X \cap Y \subseteq Y$, the hypotheses of the lemma ensure that the value of each term on the left of (14) is at least 1. Since the value of the second term on the right of (14) is exactly 1, the value of the first term on the right must be at least 1. Hence, $J \cap \partial^+(X) \neq \emptyset$.

10.5 Elimination of active circuits

Williams [Wil04] has shown that in any minimal counterexample the subgraph induced by the set of active edges is a forest:

Proposition 10.5 *No minimal counterexample has a circuit of active edges.*

PROOF: Let (G,c) be a counterexample and O a circuit whose edges are active. We show next that (G,c) is not minimal.

Let e be a minimum capacity edge in O and let $k := c_e$. Adjust the notation so that e is a forward edge of O. Let c' be the following capacity vector:

$$c'_a := \begin{cases} c_a - k & \text{if } a \text{ is forward in } O, \\ c_a + k & \text{if } a \text{ is reverse in } O, \\ c_a & \text{otherwise.} \end{cases}$$

Of course $c'_e = 0$ and therefore $N(c') \supset N(c)$. Hence, if (G,c') is a counterexample then (G,c) is not a minimal counterexample, as we promised to show. In what follows, we deal with the case where (G,c') is not a counterexample.

The sets of cuts of (G, c') and (G, c) are of course identical. Therefore, the sets of joins of (G, c') and (G, c) are identical. So, we may simply say "cut" and "join", without specifying "of (G, c')" or "of (G, c)". Notice that every cut has the same number of forward edges and reverse edges of O. Hence,

$$c'(C) = c(C) \tag{15}$$

for every cut C. Therefore,

$$\tau(G, c') = \tau(G, c). \tag{16}$$

Let $\mathcal{P}'$ be a maximum packing of joins of (G, c'). Since (G, c') is not a counterexample, $|\mathcal{P}'| = \tau(G, c')$. Let J_0 be an element of $\mathcal{P}'$. Lemma 10.3 below shows that $c'(C) - |J_0 \cap C| \geq |\mathcal{P}'| - 1$ for every cut C. So,

$$c'(C) - |J_0 \cap C| \geq \tau(G, c') - 1$$

for every cut C. By virtue of (15) and (16), this inequality holds with c in place of c', i.e.,

$$c(C) - |J_0 \cap C| \geq \tau(G, c) - 1 \tag{17}$$

for every cut C. Of course J_0 has no null edges of (G, c') and therefore no null edges of (G, c).

Now that we have a join J_0 satisfying (17), we can discard c' and $\mathcal{P}'$. Define vector c'' as follows: for each edge a,

$$c''_a := \begin{cases} c_a - 1 & \text{if } a \in J_0 \text{ and} \\ c_a & \text{otherwise.} \end{cases} \tag{18}$$

Since J_0 has no null edges, c'' is a capacity vector. The sets of cuts of (G, c'') and (G, c) are identical and therefore the sets of joins of (G, c'') and (G, c) are also identical. So, we may drop the specifications "of (G, c'')" and "of (G, c)" and say simply "cut" and "join". For every cut C, we have $c''(C) = c(C) - |J_0 \cap C|$, whence $c''(C) \geq \tau(G, c) - 1$ due to (17). Therefore,

$$\tau(G, c'') \geq \tau(G, c) - 1.$$

Let $\mathcal{P}''$ be a maximum packing of joins of (G, c''). Suppose for a moment that (G, c'') is not a counterexample. Then $|\mathcal{P}''| = \nu(G, c'') = \tau(G, c'')$. Now consider the multiset $\mathcal{P} := \mathcal{P}'' \cup \{J_0\}$ and observe that

$$|\mathcal{P}| = |\mathcal{P}''| + 1 = \tau(G, c'') + 1 \geq \tau(G, c) - 1 + 1 = \tau(G, c).$$

Notice also that $\mathcal{P}$ is a packing of (G, c), since $|\mathcal{P}(a)| = |\mathcal{P}''(a)| + 1 \leq c''_a + 1 = c_a$ for each a in J_0 and $|\mathcal{P}(a)| = |\mathcal{P}''(a)| \leq c''_a = c_a$ for each a not in J_0. Hence, $\nu(G, c) \geq |\mathcal{P}| \geq \tau(G, c)$ and therefore (G, c) is not a counterexample. This contradicts our choice of (G, c) at the start of the proof. Therefore, contrary to what we assumed for a moment, (G, c'') is a counterexample. Since $I(G'', c'') = I(G, c)$ and $N(G'', c'') \supseteq N(G, c)$ and $c'' < c$, the counterexample (G, c) is not minimal.

To finish the proof of the proposition, we must establish the following lemma:

Lemma 10.3 *For any packing $\mathcal{P}$ of joins of (G, c), any element J_0 of $\mathcal{P}$, and any cut C, the inequality $c(C) - |J_0 \cap C| \geq |\mathcal{P}| - 1$ holds.*

PROOF: Since $\mathcal{P}$ is a packing in (G, c), we have $|\{J \in \mathcal{P} : J \ni a\}| \leq c_a$ for each a in C. Hence,

$$\begin{aligned} c(C) &= \textstyle\sum_{a \in C} c_a \\ &\geq \textstyle\sum_{a \in C} |\{J \in \mathcal{P} : J \ni a\}| \\ &= \textstyle\sum_{J \in \mathcal{P}} |\{a \in C : a \in J\}| \\ &= \textstyle\sum_{J \in \mathcal{P}} |J \cap C| \,. \end{aligned}$$

Therefore, $c(C) \geq |J_0 \cap C| + \sum_{J \in \mathcal{P} \setminus \{J_0\}} |J \cap C| \geq |J_0 \cap C| + |\mathcal{P} \setminus \{J_0\}|$, since $|J \cap C| \geq 1$ for each J. It follows that $c(C) - |J_0 \cap C| \geq |\mathcal{P}| - 1$.

References

[CG02] G. Cornuéjols and B. Guenin. Note on dijoins. *Discrete Mathematics*, 243:213–216, 2002. 9

[EG77] J. Edmonds and R. Giles. A min-max relation for submodular functions on graphs. In P. L. Hammer et al., editors, *Studies in Integer Programming*, volume 1 of *Annals of Discrete Mathematics*, pages 185–204. North-Holland, 1977. 6

[FY87] P. Feofiloff and D. H. Younger. Directed cut transversal packing for source-sink connected graphs. *Combinatorica*, 7(3):255–263, 1987. 4, 7

[LY78] C. L. Lucchesi and D. H. Younger. A minimax theorem for directed graphs. *J. of the London Math. Soc. (2)*, 17:369–374, 1978. 3, 5

[Sch80] A. Schrijver. A counterexample to a conjecture of Edmonds and Giles. *Discrete Math.*, 32:213–214, 1980. 7

[Sch82] A. Schrijver. Min-max relations for directed graphs. *Annals of Discrete Math.*, 16:261–280, 1982. 4, 7

[Sch03] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, 2003. 1, 2, 4, 5

[WG05] A. M. Williams and B. Guenin. Advances in packing directed joins. *Electronic Notes in Discrete Mathematics*, 19:249–255, 2005. https://www.sciencedirect.com/science/journal/15710653. 11

[Wil04] A. M. Williams. Packing directed joins. Masters Thesis, University of Waterloo, 2004. 1, 9, 11, 16

[Woo78a] D. R. Woodall. Menger and Kőnig systems. In Y. Alavi and D. R. Lick, editors, *Theory and Applications of Graphs*, volume 642 of *Lecture Notes in Mathematics*, pages 620–635. Springer, 1978. 2

[Woo78b] D. R. Woodall. Minimax theorems in graph theory. In L. W. Beineke and R. J. Wilson, editors, *Selected Topics in Graph Theory*, pages 237–269. Academic Press, 1978. 2