Functions on the sphere with critical points in pairs and orthogonal geodesic chords

RISM4 – Nonlinear Phenomena
in Mathematics and Economics

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Outline of this talk.

Main topics

Abstract

I will discuss a problem of multiplicity for geodesics starting and arriving orthogonally to the boundary of a Riemannian ball using Morse theory. This gives an analogous multiplicity result for a class of periodic solutions (brake orbits) in a potential well of a Lagrangian system.

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3. Geometry: orthogonal geodesic chords

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The setup:

- M^m is a compact manifold;
- $\beta_k(M)$ denotes the k-th Betti number of M, $k = 0, \ldots, m$;
- $f\colon M \to \mathbb{R}$ is a Morse function;
- if $p \in M$ is a critical point of f, $\mathrm{i}_{\mathrm{Morse}}(f, p)$ is the Morse index;
- $\mu_k(f)$ is the number of critical pts of f having Morse index equal to k

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Definition

f is *Morse-even* if $\mu_k(f)$ is even for all $k = 0, \ldots, m$.

Proposition

If $f\colon \mathbb{S}^m \to \mathbb{R}$ is Morse-even, then $\mu_k(f) > 0$ for all $k = 0, \ldots, m$.

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$\mu_2 \geq \mu_1 - \mu_0 + \beta_2 - \beta_1 + \beta_0 = \mu_1 - \mu_0 + 1 > 0$.... □

Theorem

If M^m is a compact manifold which is connected and orientable ($\beta_0(M) = \beta_m(M) = 1$) with $\beta_k(M) \in 2\mathbb{N}$ for all $k = 1, \dots, m-1$, and $f\colon M \to \mathbb{R}$ is a Morse-even function, then:

$$\mu_k(f) > \beta_k, \quad \text{for all } k = 0, \dots, m.$$

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$$M = \underbrace{(\mathbb{S}^2 \times \mathbb{S}^2)\sharp \cdots \sharp(\mathbb{S}^2 \times \mathbb{S}^2)}_{k \text{ times}} \ \sharp \ \underbrace{\mathbb{C}P^2 \sharp \cdots \sharp \mathbb{C}P^2}_{(2m) \text{ times}}$$

Conservative Lagrangian systems:

- (M, g) Riemannian manifold (configuration space)
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Special class of periodic solutions: brake orbits (pendulum-like)

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Geometric construction:

- remove from Ω_E a suitably defined neighborhood V of $\partial\Omega_E$;
- geodesics in Ω_E with endpoints in $\partial\Omega_E$ correspond to geodesics in $\Omega' = \Omega_E \setminus V$ arriving orthogonally to $\partial\Omega'$
- Ω' is homeomorphic to $\Omega_E \cong B^{m+1}$
- $\partial\Omega' \cong \mathbb{S}^m$ is concave.

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Basic assumptions on (Ω, g)

For all $p \in \partial\Omega$:

(HP1) $\exists T_p > 0$ such that:

- $\gamma_p(t) \notin \partial\Omega$ for $t \in]0, T_p[$;
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(HP2) $\gamma_p(T_p)$ is not a focal point along γ_p.

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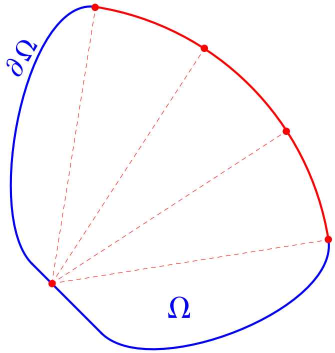
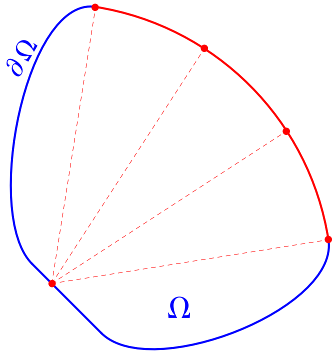
- (HP1) is an open condition relatively to the C^1-topology
- (HP2) is an open condition relatively to the C^2-topology
- Radially symmetric metrics on balls satisfy (HP1) and (HP2)
- Neither (HP1) nor (HP2) is generic.

Critical points of the crossing time



Obs. 1: By transversality (HP1), $T\colon \partial\Omega \longrightarrow \,]0,+\infty[$ is smooth.



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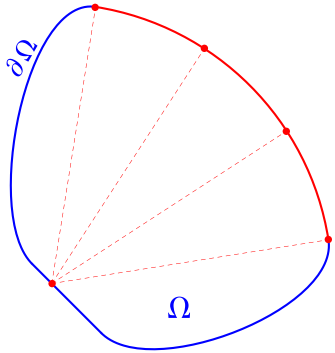

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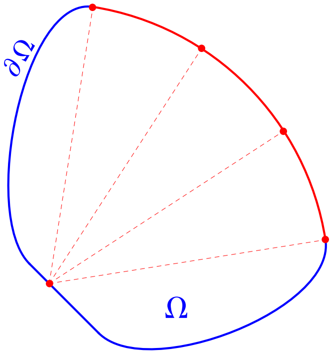
Under assumption (HP2), p is a critical point of T iff γ_p is an orthogonal geodesic chord, i.e., iff $\dot\gamma_p(T_p) \perp \partial\Omega$.



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Obs. 2: Critical points of $T : \partial\Omega \to \mathbb{R}$ come in pairs!

Morse indices associated to an OGC

- $\gamma_p \colon [0, T_p] \longrightarrow \overline{\Omega}$ orthogonal geodesic chord.
- $\gamma_p(0) = p$, $\gamma_p(T_p) = q$
- $\gamma_q = \gamma_p^-$ (backward reparameterization)

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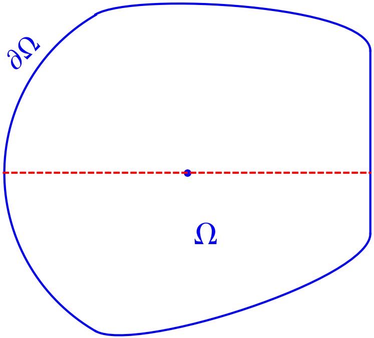
The crossing time is a Morse-even function

In general, the number focal points depends on the orientation!



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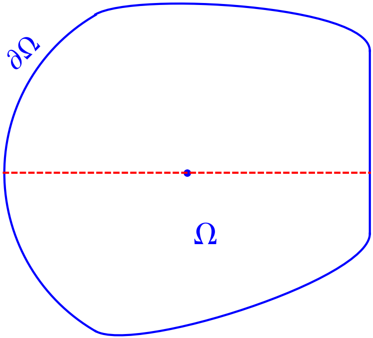

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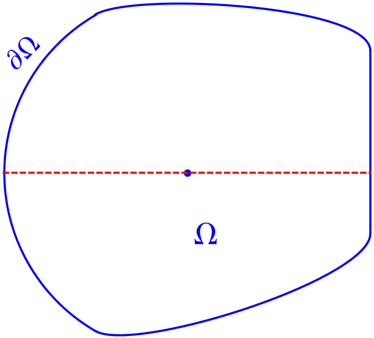

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Proof.

- Stability of the focal points
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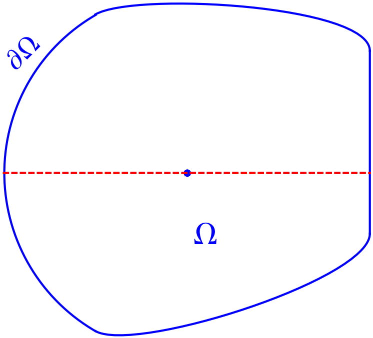

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Obs.: Example shows that (HP2) is not generic.

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This settles Seifert's conjecture in a quite large number of cases.

Counterexample when $\partial\Omega$ is not a sphere

When $\partial\Omega$ is not connected, one cannot expect the existence of more than 2 OGC's, regardless of the dimension.