What Is Geometry? The 1982 Presidential Address
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What is geometry?

The 1982 Presidential address

MICHAEL ATIYAH

1. History

Of all the changes that have taken place in the mathematical curriculum, both in schools and universities, nothing is more striking than the decline in the central role of geometry. Euclidean geometry, together with the allied subject of projective geometry, has been dethroned and in some places almost banished from the scene. While educational reform was certainly needed there is always the danger that the pendulum may swing too far the other way and that insufficient attention may be paid to geometry in its various forms. Much of the difficulty here centres round the elusive nature of the subject: What is geometry? I would like to examine this question in a very general way in the hope that this may clarify the educational reasons for teaching geometry, and for deciding what is appropriate material at different levels.

Let me begin by taking a historical look at the development of mathematics. It is I think no accident that geometry, in the hands of the Greeks, was the first branch of mathematics to reach maturity. The fundamental reason is that geometry is the least abstract form of mathematics: this means that it has direct applicability to everyday life and also that it can be understood with less intellectual effort. By contrast algebra is the essence of abstraction, involving a dictionary of symbolism which has to be mastered by great effort. Even arithmetic, based as it is on the process of counting, depends on its own dictionary such as the decimal system, and took longer to evolve.

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Of course, at a sophisticated level, geometry does involve abstraction. As the Greeks recognised, the points and lines which we meet in the real world are only approximations to some "ideal" objects, in an "ideal" world where points have no magnitude and lines are perfectly straight. These philosophical reflections do not however worry the practitioner of geometry, be he a school-child or a civil engineer and geometry at this level remains the practical study of physical shapes.

For many centuries Euclidean geometry dominated the mathematical scene, but the emergence of algebra, its application by Descartes to geometry and the subsequent development of calculus altered the whole character of mathematics. It became much more symbolic and abstract. Inevitably geometry came to be regarded as primitive and old-fashioned.

While these rival branches of mathematics were being developed the foundations of geometry and its relation to the physical world were being re-examined. In the nineteenth century the famous "parallel postulate" of Euclid, asserting the existence of a unique line parallel to a given line and passing through a given point, was shown to be independent of the other axioms. Non-Euclidean geometries, in which this postulate does not hold, were discovered. This had the profound if disturbing effect of liberating geometry from physics. While there is only one physical universe there are many different geometries and it is not clear which one is most relevant to our universe. For a while algebra attempted to take advantage of this division of ranks in the geometrical field. Felix Klein in his famous *Erlanger Programm* attempted to define geometry as the study of those properties which are invariant under a given group of symmetries: different geometries corresponding in this way to different symmetry groups. While this has been a very fruitful point of view in connection with non-Euclidean geometries its scope had already been undermined by the earlier far-reaching ideas of Bernhard Riemann. For Riemann space did not have to be homogeneous, its curvature could vary from point to point and there might be no symmetries at all. Instead of group theory Riemann based his geometry on the differential calculus and, as we know, his view-point was ultimately vindicated by Einstein's general theory of relativity.

The outcome of all this introspection by geometers showed that geometry is not just the study of physical space. In particular it is not restricted to 3 (or 4) dimensions. But in that case what is the use of all these other geometries? Are they simply abstract games played by mathematicians? I shall attempt to answer these questions by showing how abstract "spaces" and geometries occur naturally in a wide variety of ways. Some of my examples will be blindingly familiar but others may be more novel.

2. *Examples of non-physical spaces*

Graphs. At all levels in mathematics graphs are familiar and widely used. The simplest is perhaps the distance/time graph for a moving object. Of

course the (x,t) plane is part of Einstein's 4-dimensional space-time, but in our graph the time variable t is replaced by a second space variable y. For other examples, however, such as those used in economics, the variables may have no relation to space-time at all. The plane in which we draw our graph is an abstract plane, but the practical advantage of such pictorial representations is clearly enormous. The advantage rests on the capacity of the brain to see two-dimensional patterns literally at a glance.

The complex plane. The representation of complex numbers $x + iy$ by points of a plane is again an almost too familiar example of an "abstract" plane. If for instance x represents distance along a fixed line, and if an algebraic problem involving this distance leads to a complex solution $x + iy$, then y does not correspond to any real direction. Familiarity breeds contempt and long exposure to complex numbers makes the complex plane almost tangible. However, as the great Gauss said "the true metaphysics of $\sqrt{-1}$ is elusive". Those who have to introduce students to complex numbers for the first time may well agree.

Riemann surfaces. While the two previous examples are familiar, when we combine them the geometry becomes more serious. Consider for example the graph of the (two-valued) function $y^2 = f(x)$ where f is some polynomial. When x,y are real we can draw this in an ordinary real plane, but if we take x and y both to be complex the graph becomes a real surface in a four-dimensional real space. This is the Riemann surface of the function and its geometric (or topological) properties are of fundamental importance in the analytic study of the function. This illustrates the importance of "abstract" geometrical ideas in the study of polynomials or analytic functions of several variables. In fact complex algebraic (and analytic) geometry are now flourishing branches of mathematics.

Dynamics. In Newtonian mechanics the motion of a particle in a given field of force is determined if we know its position and velocity at one instant. To describe its subsequent motion it is then convenient to introduce the "phase-space" of pairs (x,v), where both components are 3-vectors and represent position and velocity respectively. The motion will then be represented by a curve $((x(t),v(t))$ in this 6-dimensional space. For instance if motion is on a line rather than in 3-space, the phase space is then 2-dimensional and simple harmonic motion corresponds to circles in this phase-plane. These phase pictures are extremely useful in general dynamical problems.

Rigid bodies. Suppose, instead of a particle, we have a rigid body. Before proceeding to look at motion of such a body consider first the static problem of just describing its position. Assuming its centre of mass fixed at the origin we are left with the rotations about the origin. Such rotations have three

degrees of freedom but these do not correspond to specifying 3 Cartesian coordinates (x,y,z). The "space" of rotations is actually the Non-Euclidean (elliptic) 3-space, the quotient of the 3-sphere by the anti-podal map. (The most elegant way of seeing this is to use quaternions. The quaternions q of unit norm form the 3-sphere and act on the 3-space of imaginary quaternions x by $q \times q^{-1}$. This gives all rotations of 3-space but $\pm q$ give the same rotation and are anti-podal pairs on the 3-sphere.) Thus non-Euclidean geometry appears here out of Euclidean geometry.

Line geometry. If our rigid body is replaced by a long thin rod idealised to a line of infinite length then we cannot describe its position in quite the same way because its centre of mass is not now defined. Instead we can pick two points $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ on the line and consider the 6-vector $(X - Y, X \wedge Y)$. If we pick any other pair on the line this 6-vector gets multiplied by a scalar. Moreover its components satisfy the quadratic relation

$$(X - Y).(X \wedge Y) = 0$$

This means that lines in 3-space can be parameterised by points of a quadric in projective 5-space. This is the famous Klein representation. When I first encountered it as a young student I thought it one of the most beautiful ideas in mathematics. To illustrate its properties let me recall that in 3-space a hyperboloid of one sheet has two systems of generating lines. Similarly in 5-space the Klein quadric has two systems of generating planes. A plane of one system parametrises all lines in 3-space through a fixed point, while a plane of the other system parametrises all lines in 3-space lying in a fixed plane. From this we can immediately deduce the incidence properties of the generating planes. Namely two planes of the same system always meet in a point (e.g. because two points in 3-space lie on a unique line), while planes of opposite systems usually do not meet (because there is in general no line in 3-space which goes through a given point p and lies in a given plane π. However exceptionally planes of opposite systems can meet and then they meet in a line (if P lies in π there is a whole pencil of lines through P lying in π).

It is interesting that this Klein representation has in recent years played a fundamental role in the work of Roger Penrose in theoretical physics. Roughly speaking Penrose thinks of the Klein quadric as space-time (after complexification) and the original 3-space (also complexified) is then a fundamental auxiliary space (called twistor space) which is supposed to be more basic in some ways than space-time (e.g. it has fewer dimensions).

Function spaces. If instead of a rigid rod we now have a finite piece of string then its possible positions in 3-space will require infinitely many parameters. We can describe its position by three functions $x(t)$, $y(t)$, $z(t)$ where t is a parameter on the string (e.g. distance from one end). The "space" of all

positions of the string is therefore an infinite-dimensional space. Such function-spaces occur frequently in calculus of variations problems (when we are trying to minimise some quantity depending on a function). Geometrical ideas have proved very useful in these contexts, notably in connection with "fixed-point theorems".

3. Conclusions

The preceding examples were designed to illustrate the fact that spaces, frequently of high dimension, arise quite naturally out of realistic situations. I chose examples from 3-dimensional mechanics to emphasise this reality. Of course a sophisticated mathematician will quite happily start with n real variables $(x_1, \ldots x_n)$ and regard these as coordinates of a point in n-dimensional space. Such an abstract starting point may not however be entirely convincing to the sceptic who may have serious doubts about the "meaning" of higher-dimensional geometry.

Are we now any further forward towards answering the initial question "What is geometry?"? If geometry is not just the study of physical space but of any abstract kind of space does this not make geometry coincide with the whole of mathematics? If I can always think of n real variables as giving a point in n-space what distinguishes geometry from algebra or analysis?

To get to grips with this question we have to appreciate that mathematics is a human activity and that it reflects the nature of human understanding. Now the commonest way of indicating that you have understood an explanation is to say "I see". This indicates the enormous power of vision in mental processes, the way in which the brain can analyse and sift what the eye sees. Of course, the eye can sometimes deceive and there are optical illusions for the unwary but the ability of the brain to decode two- and three-dimensional patterns is quite remarkable.

Sight is not however identical with thought. We have trains of thought which take place in sequential form, as when we check an argument step by step. Such logical or sequential thought is associated more with time than with space and can be carried out literally in the dark. It is processes of this kind which can be formalised in symbolic form and ultimately put on a computer.

Broadly speaking I want to suggest that geometry is that part of mathematics in which visual thought is dominant whereas algebra is that part in which sequential thought is dominant. This dichotomy is perhaps better conveyed by the words "insight" versus "rigour" and both play an essential role in real mathematical problems.

The educational implications of this are clear. We should aim to cultivate and develop both modes of thought. It is a mistake to overemphasise one at the expense of the other and I suspect that geometry has been suffering in recent years. The exact balance is naturally a subject for detailed debate and must depend on the level and ability of the students involved. The main point

I have tried to get across is that geometry is not so much a branch of mathematics as a way of thinking that permeates all branches.

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Digital roots, rings and clock arithmetic

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The 'digital root' of 73219 is 4, since $7 + 3 + 2 + 1 + 9 = 22$ and $2 + 2 = 4$. This idea features in some popular puzzles (and in a recent *Gazette* note **65.26**). Section 1 of this article concerns some basic properties of digital roots, while Section 2 gives a characterisation of functions like the 'digital root' function. The former provides interesting exercises in modular arithmetic; the latter in elementary ring theory.

1. Elementary properties

Let N be any positive integer. Then we may express N in base k,

$$N = a_0 + a_1 k + a_2 k^2 + \cdots + a_n k^n$$

where each a_i lies between 0 and $k - 1$. To calculate the digital root (base k) of N.

> first add up the (base k) digits $a_0, a_1, \ldots, a_n$ of N
> and express the answer N_1 in the base k,
> then add up the (base k) digits of N_1 and express the answer N_2 in base k,

and so on, continuing in this way until we obtain a single digit number (base k), which we then call the *digital root* (base k) of N and denote by d.r.(N).

For simplicity we shall concentrate in this section on digital roots (base 10); for the corresponding results (and proofs) in base k one simply replaces 10 by k, and 9 by $k - 1$ throughout.

The first fact about digital roots (base 10) is so obvious that one could almost overlook it, namely

(1) for each positive integer N, d.r.(N) $\in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

When combined with the next fact, this simple observation provides a very quick way of calculating the digital root of any positive integer N.

(2) d.r.(N) $\equiv N \pmod 9$.