Algebras, quivers and adjunctions

Plan

1. Concept of adjoint functors.
2. Basic fin.-dim. algebras and their quivers. Path algebra of a quiver.
3. Adjunction in 2.

1.

"I didn't invent ~~invent~~ cat. theory to talk about functors. I invented it to talk about natural transf".
S. Mac Lane.

a) ¿What is natural transformation?

It's a map from one functor to another!!

Ex. Let V - vector space. We have "natural" map

$$\varphi_V : V \to V^{**}$$
$$v \mapsto (f \mapsto f(v))$$

More over if $\dim V < \infty \Rightarrow \varphi_V$ is iso!.
What naturality of φ_V means?. If $\dim V < \infty$, $V \simeq V^*$
But there is not canonical way to construct it.
Any $L : U \to V$ gives rise to linear map
$L^{**} : U^{**} \to V^{**}$, and we have the following commutative (?) diagram.

~~(f~~

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\varphi_{\mathcal{U}}"] \arrow[d, "L"'] & \mathcal{U}^{**} \arrow[d, "L^{**}"] \\ V \arrow[r, "\varphi_V"] & V^{**} \end{tikzcd}$$

If $u \in \mathcal{U}$, then $\boxed{\varphi_V(L(u)) = L^{**}(\varphi_{\mathcal{U}}(u))}$?

Left side: If Any $f \in V^*$, then

$$\varphi_V(L(u))(f) = f(L(u))$$

Right side: For any $f \in V^*$, we have

$$\left(L^{**}(\varphi_{\mathcal{U}}(u))\right)(f) = \varphi_{\mathcal{U}}(u)(L^*(f)) = L^*(f)(u) = f(L(u))$$

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Def. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two covariant functors

Natural transformation $\alpha : F \to G$ is a rule which for any $X \in \mathcal{C}$ associate a morfism

$$\alpha_X : F(X) \to G(X),$$ such that

for any $f : X_1 \to X_2$ in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd} F(X_1) \arrow[r, "\alpha_{X_1}"] \arrow[d, "F(f)"'] & G(X_1) \arrow[d, "G(f)"] \\ F(X_2) \arrow[r, "\alpha_{X_2}"] & G(X_2) \end{tikzcd}$$

If α_X - is iso, the α - is natural isomorfism.

Example (above)

Let $\mathcal{C} = Vect_k$, $\mathcal{D} = Vect_k$.

$F: Vect_k \to Vect_k$ — identity functor.

$G: Vect_k \to Vect_k$ — double duality

$\left(\begin{array}{l} V \longmapsto V^{**} \\ (L: U \to V) \longmapsto (L^{**}: U^{**} \to V^{**}) \end{array} \right)$

Then $U \longmapsto \varphi_U$ — defines a natural transformation between functors.

(6) What is adjunction? ("Adjoint functors arise every where", Mac Lane).

Ex, Let $\mathcal{C} = Sets$, $\mathcal{D} = Vect_k$.

$F: Sets \longrightarrow Vect_k$ — free functor

$X \longmapsto span\, X = \{ \sum \lambda_i x_i \mid \lambda_i \in k, x_i \in X \}$

$G: Vect_k \longrightarrow Sets$ — forgetfull functor.

$V \longrightarrow V$

Any function $g: X \to G(V)$ uniquely extended to a linear map, $f: F(X) \longrightarrow V$, by

$f(\sum \lambda_i x_i) = \sum \lambda_i g(x_i)$. This corespondence

$\eta: g \longmapsto f$ has an inverse $\mu: f \longmapsto f|_X$

which for any linear map $f: F(X) \to V$ gives a function $f|_X : X \to G(V)$ restricting on the basis X therefore $\eta = \eta_{X,V}$ defines a bijection

$$\eta : \underline{Sets}(X, G(V)) \simeq \underline{Vect_k}(F(X), V).$$

Moreover it is defined "naturaly" for any X and V.

<u>Def.</u> Let $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ be two covariant functors. <u>Adjunction</u> between F e G is a bijection

$$\eta_{A,B} : \mathcal{C}(A, G(B)) \simeq \mathcal{D}(F(A), B)$$

for any pair $(A,B) \in \mathcal{C} \times \mathcal{D}$, which is "natural" in A, and B. Which means that

For any $f: A \to A'$ we have

$$\begin{tikzcd}
\mathcal{C}(A, G(B)) \arrow[r, "\eta_{A,B}"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}(F(A), B) \arrow[d, "(F(f))^*"] \\
\mathcal{C}(A', G(B)) \arrow[r, "\eta_{A',B}"] & \mathcal{D}(F(A'), B)
\end{tikzcd}$$

and

For any $g: B \to B'$

$$\begin{tikzcd}
\mathcal{C}(A, G(B)) \arrow[r, "\eta_{A,B}"] \arrow[d, "G(g)_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}(F(A), B) \arrow[d, "g_*"] \\
\mathcal{C}(A, G(B')) \arrow[r, "\eta_{A,B'}"'] & \mathcal{D}(F(A), B')
\end{tikzcd}$$

In this case F – left adjoint to G
G – right adjoint to F.

Rem. 1 In example above
F – is left adjoint to G.
G – is right adjoint to F.

Rem. 2. Let $A : U \to V$ – linear maps between complex vector spaces.
B, ~~$V^* \to U^*$~~
B is called adjoint to A if
$\langle x, Ay \rangle = \langle Bx, y \rangle$ for any $x, y \in V$.

Examples (of adjunction).

a) "free – forgetfull" functors
left adjoint right adjoint

F: $\underline{Sets} \longrightarrow (\underline{Groups})\ (\underline{Rings}), \underline{R\text{-mod}}, \ldots,$
$X \longrightarrow F\langle X \rangle$

$G : \underline{Groups} \longrightarrow \underline{Sets}$ (forgetfull functor).

b) Let $\mathcal{C} = \underline{Top}$ (category of topological vector spaces, morfisms – countinious maps).

$F : \underline{Top} \longrightarrow \underline{Sets}$ has both left and right adjoints
Largest topology

$L : \underline{Sets} \to \underline{Top}$,
$X \mapsto L(X)$, $=$ X as a set with discrete topology. $=$ "all subsets" are open

Therefore any map $L(x) \to Y$ are continious
$\underline{Sets}(X, F(Y)) = \underline{Top}(L(X), Y)$.

$R: \underline{Sets} \to Top$

$X \to R(X)$ – some set, with trivial topology

"smallest topology"

two open sets: X and $\emptyset$.

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c) Let I be a poset.

Then $\mathcal{C}_I$ – is a category:

<u>Objects</u> Elements $i \in I$,

<u>Morfisms</u>: Given two $i, j \in I$ we have

$$Mor(i,j) = \begin{cases} \emptyset, \text{ if } & i \nleq j \\ \text{one element} & \text{if } i \leq j \end{cases}$$

Given two posets I and J, any functor $F: \mathcal{C}_I \to \mathcal{C}_J$ order-preserving map $F: I \to J$

Therefore an adjunction is a pair of order-preserving functions

$$F: \mathcal{C}_I \to \mathcal{C}_J$$
$$G: \mathcal{C}_J \to \mathcal{C}_I$$
such that

$$\mathcal{C}_I(F(a), b) \simeq \mathcal{C}_J(a, G(b))$$

for any $a \in I$ and $b \in J$, which means that

$$F(a) \leq_J b \iff a \leq_I G(b)$$

<u>For instanse</u>: I – poset of all ideals in $\mathbb{C}[x_1, \dots, x_n]$ – commutative ring

J – poset of all sub sets in $\mathbb{C}^n$.

$f: I \to J^{op}$

ideal of function $\longmapsto$ zero set of ideal

(~~Con~~ covariant functor) $g: J^{op} \to I$

$\{\text{set of points } X\} \to \{\text{ideal of functions which vanish on } X\}$

(f, g) – is adjoint pair.

d) Let $\mathcal{C} = \underline{\text{AbGroup}}$ $\mathcal{D} = \underline{\text{Group}}$

$\mathbb{F} : \underline{\text{AbGroup}} \longrightarrow \underline{\text{Group}}$ (forgetfull functor).

$\text{Ab}: \underline{\text{Group}} \longrightarrow \underline{\text{AbGroup}}$

$G \longmapsto G/_{[G,G]}$

Exer. Show that Ab is left adjoint to $\mathbb{F}$.

e) Standart fact in linear algebra tell us that

$$\text{Hom}(U \otimes V, W) \cong \text{Hom}(U, \text{Hom}(V, W))$$

in which U, V, W, - vector spaces
$\text{Hom}(-,-)$ - the set of linear maps.

Similarly for R-mod

$$\text{Hom}_S(Y \otimes_R X, Z) \cong \text{Hom}_R(Y, \text{Hom}_S(X, Z))$$

In which X, - R-S - bimod
$Y \in$ R-mod, $Z \in$ S-mod.

Therefore $- \otimes_R X$ - is left adjoint to $\text{Hom}_S(X, -)$

Particular case:

G - finite group, $\mathbb{C}[G]$ - its group algebra

H - subgroup of G, $\mathbb{C}[H]$ — " — " — " —

~~$\text{Hom}_{\mathbb{C}[H]}(X, Y \otimes_R X, Z) \text{Hom}$~~

We have Frobenious reciprocity formula

$$\text{Hom}_{\text{Rep}(H)}(\text{Res}^G_H W, U) \simeq \text{Hom}_{\text{Rep}(G)}(\text{Ind}^G_H W, U)$$

Theorem Specifying the adjunction between 2 functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ is equivalent to specifying 2 natural transformations $\eta: Id_{\mathcal{C}} \to GF$ and $\varepsilon: FG \to Id_{\mathcal{D}}$ such that

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, "\varepsilon F"] \\ & {} \end{tikzcd}, \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, "G\varepsilon"] \\ & {} \end{tikzcd}$$

η – identity of adjunction

ε – co identity.

Ex. let $\begin{tikzcd} \underline{Sets} \arrow[r, bend left, "F"] & \underline{Vect}_k \arrow[l, bend left, "G"] \end{tikzcd}$

for any $X \in \underline{Sets}$, $\eta_X: X \to GF(X)$ (is inclusion of basis)

$V \in \underline{Vect}_k$, $\varepsilon_V: FG(V) \to V$ (extension of identity map).