Duflo isomorfism

① Statement

Let $\mathbb{K}$ - ~~algebraically closed~~ field, char $k=0$.

$\mathfrak{g}$ — Lie algebra over $\mathbb{K}$, $\dim \mathfrak{g} < \infty$.

Let $S(\mathfrak{g})$ — its symmetric algebra, i.e. given a basis $x_1, \dots, x_n$ in $\mathfrak{g}$, $S(\mathfrak{g}) = \mathbb{K}[x_1, \dots, x_n]$. of commutative variables $x_1, \dots, x_n$.

$\mathcal{U}(\mathfrak{g})$ — universal enveloping algebra, i.e.

$\mathcal{U}(\mathfrak{g}) = \mathbb{K}\langle x_1, \dots, x_n \mid [x_i, x_j] = c_{ij}^k x_k \rangle$.

Using PBW theorem we have that

$$I_{PBW}: S(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$$

$$x_1 \odot \dots \odot x_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}$$

is an isomorfism of filtered vector spaces $S(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$.

$\mathfrak{g}$ acts on $\mathcal{U}(\mathfrak{g})$ by ad-action:

$$ad_x(u) = xu - ux, \quad x \in \mathfrak{g}, \; u \in \mathcal{U}(\mathfrak{g})$$

$\mathfrak{g}$ — // — // $S(\mathfrak{g})$ by extension of ad action on $\mathfrak{g}$

$$ad_x(y^n) = n \cdot [x, y] \cdot y^{n-1}, \quad x, y \in \mathfrak{g}, \quad y^n \in S(\mathfrak{g})$$

or given a monomial $y_1 \odot \dots \odot y_n \in S(\mathfrak{g})$

$$ad_x(y_1 \odot \dots \odot y_n) = \sum_i y_1 \odot \dots \odot [x, y_i] \odot \dots \odot y_n.$$

Exercise (home work) show that

$$\begin{tikzcd} S(g) \arrow[r, "ad_x"] \arrow[d, "I_{PBW}"'] \arrow[dr, phantom, "\circlearrowleft"] & S(g) \arrow[d, "I_{PBW}"] \\ U(g) \arrow[r, "ad_x"] & U(g) \end{tikzcd}$$

, or $ad_x \circ I_{PBW} = I_{PBW} \circ ad_x$

$x \in g$

Hence we have that $I_{PBW} : S(g)^g \longrightarrow U(g)^g$.

is a vector space isomorfism.

Remark
- $U(g)^g = Z(U(g))$ — is a center of $U(g)$.
- I_{PBW} is not (!) an algebra isomorfism.

Theorem (Duflo, 74'). $Duf = I_{PBW} \circ J^{1/2}$ — is an isomorfism

$$J(x) = \det\left(\frac{1-e^{-ad_x}}{ad_x}\right) \text{ — Duflo element.}$$

Example $g = su(2) = \langle x, y, z \rangle$ — real Lie algebra

$[x,y] = z$, $[y,z] = x$, $[z,x] = y$

In this case $(S(g))^g = \mathbb{R}[x^2+y^2+z^2]$.

for instance $ad_x(x^2+y^2+z^2) = 2[x,y]\cdot y + 2[x,z]\cdot z =$

$= 2zy - 2yz = 0$ in $S(su(2))$.

$U(g)^g = Z(U(su(2))) = x^2+y^2+z^2$ — Casimir element.

$Duf: x^2+y^2+z^2 \longmapsto x^2+y^2+z^2+\frac{1}{4}$.

Remarks.

① If char $\Bbbk \neq 0$ — statement fails.

For instance: $g = sl(2)$ over field $\Bbbk$ with char $\Bbbk = p > 3$.

"Proof" $Z(g)$ is $\Bbbk < E = e^p, F = f^p, H = h^p - h, \overset{\to \text{Casimir}}{C} \mid$

$4EF + H^2 = C^p - 2C^{(p+1)/2} + C >$

For other side $S(g)^g$ given by $X = e^p, Y = f^p, Z = H^p$

$\Omega = 4ef + h^2$, subject to

$4XY + Z^2 = \Omega^p$.

Specm $S(g)^g$ has 1 singular point.

Specm $Z(g)$ —//— $(p-1)/2 > 1$ singular point.

Hence they cannot be isomorfic. "$\square$".

② It is not unique (!).

③ It coincides with Harish-Chandra isomorfism:

Let g be semi-simple,

$g = n^- \oplus h \oplus n^+$ - Cartan decomposition.

$U(g) = U(h) \oplus (U(g) n^+ + n^- U(g))$ not direct (!).

Exercise $z \in Z(g) \Rightarrow z \in U(h) \oplus (U(g) n^+ \cap n^- U(g))$.

Let $\gamma : U(g) \longrightarrow U(h)$ - projection on first factor

$\cong$

$S(h)$.

Let $\rho = \frac{1}{2}\sum_{\alpha_i \in \Delta^+} \alpha_i$, we consider the map

$$\tau : \mathfrak{h} \longrightarrow S(\mathfrak{h})$$
$$H \longmapsto H - \rho(H)\cdot 1$$

which extends to automorfism $S(\mathfrak{h}) \to S(\mathfrak{h})$.

<u>Theorem</u> (Harish-Chandra)

$$\tilde{\gamma} = \gamma \circ \tau : \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^{W} \cong S(\mathfrak{g})^{\mathfrak{g}}$$

is an isomorfism of algebras.

$S(\mathfrak{h})^{W}$ — invariant element w.r.t. Weyl group W.

<u>Example</u> $\mathfrak{g} = sl_2$ $\quad Z(sl_2) \simeq \mathbb{R}[\Omega]$

x, y, h $\quad \Omega = \frac{1}{2}H^2 + XY + YX$ — Casimir element.

$[h,x] = 2x$, $[h,y] = -2y$, $[x,y] = h$

$xy = [x,y] + yx = h + yx$. hence

$$\Omega = \frac{1}{2}h^2 + h + yx, \text{ and } \gamma(\Omega) = \frac{1}{2}h^2 + h.$$

$\Delta^+(sl_2) = \{\alpha\}$. $W_{sl_2} = \{\pm\alpha, h \to -h\}$

$\Rightarrow \quad \tau : h \to h - \rho(h)\cdot 1 = h - \frac{1}{2}\alpha(h)\cdot 1 = h - 1.$

$\Rightarrow \quad \tilde{\gamma} = \gamma \circ \tau = \frac{1}{2}(h-1)^2 + (h-1) = (h-1)\cdot\left(\frac{1}{2}(h-1) + 1\right) = \frac{1}{2}(h^2 - 1)$

W-invariant //

② Understanding (Duflo element).

ⓐ Bernoulli numbers.

Consider the sum: Jacob Bernoulli

$$1^m + 2^m + \ldots + n^m \overset{\swarrow}{=} \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} \cdot B_k \cdot n^{m+1-k}$$

In which

$B_0 = 1$, $B_1 = +1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$ ... $B_{2k+1} = 0$, $k \geq 1$.
$B_6 = 1/42$, ...

Generating function:

$$\frac{t}{e^t - 1} \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \text{ or } \frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{k=2}^{\infty} B_k \frac{t^k}{k!}.$$

For instanse it is well-known.

$$1 + 2 + \ldots + n = \frac{1}{2}(n^2 + n) = \frac{1}{2}(B_0 n^2 + 2B_1 n^1).$$

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right) = \frac{1}{3}(B_0 n^3 + 3B_1 n^2 + 3B_2 n^1).$$

Modified Bernolli numbers:

$$b_{2k} = \frac{B_{2k}}{(4k)\cdot(2k)!}$$

they generating function is $\sum_{k=0} b_{2k} \cdot t^k = \frac{1}{2} \ln\left(\frac{sh\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)$

with $sh\left(\frac{x}{2}\right) = \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}$ – sinus hyperbolic.

Now "using" mnemonic rule:

$$\det e^M = e^{\text{tr} M}, \text{ or } \det M = \exp(\text{tr}(\ln M)).$$

Hence we have that

$$J^{1/2}(x) = \left(\det\left(\frac{e^{(ad_x)/2} - e^{-ad_x/2}}{ad_x}\right)\right)^{1/2} =$$

$$= \left(\exp\left(tr\left(\ln\left(\frac{sh(ad_x/2)}{ad_x/2}\right)\right)\right)\right)^{1/2} =$$

$$= \exp\left(\frac{1}{2} tr(ad_x) + \frac{1}{2}\sum_{k=2}^{\infty} \frac{B_k}{k \cdot k!} tr(ad_x^k)\right)$$

$tr(ad_x^k) \in S(g^*)$ – element of dual Lie-algebra,

$SS = \frac{1}{2} tr(ad_x) + \frac{1}{2}\sum_{k=2}^{\infty} \frac{B_k}{k \cdot k!} tr(ad_x^k) \in \hat{S}g^*$ – formal power series on g^*.

So think of ~~Duflo~~ element $J(x)$ as a certain serie $J^{1/2}(x) = \sum_{i=0}^{\infty} a_k \, tr(ad_x^k)$

<u>claim</u> $tr(ad_x^k)$ is g-invariant for any $k \in \mathbb{N}$.

$\Rightarrow J^{1/2}(x) \in \hat{S}(g^*)^g$

Now we have to explain how $\hat{S}(g^*)$ acts on $S(g)$. First any $\xi \in g^*$ acts on $S(g)$ by derivation, given $x \in g$ we have

$$\xi \cdot x^n = n \cdot \xi(x) \cdot x^{n-1}$$

By extension $(\xi^k) \in S^k(g^*)$ acts as follows

$$(\xi^k).\, x^n = n \cdots (n-k+1)\, \xi(x)^k .\, x^{n-k}.$$

Hence $\hat{S}(g^*)$ acts $S(g)$. And $\hat{S}(g^*)^g$ acts on $S(g)^g$.

So everything is well-defined.

"Proof"

Duflo 77 → Ivan

Kontsevich 97. → Formality theorem.

Torossian 08 → Drinfeld associators

③ Kashiwara – Vergne "conjecture":

Let G – Lie group
g – its Lie algebra.

$\exp: g \to G$ – exponential map.
$\log: G \to g$ – "inverse".

If X & Y commutes in g then

$$\log(e^X \cdot e^Y) = X+Y$$

Problem: Find $\log(e^X \cdot e^Y)$ in terms of X & Y
if $[X,Y] \neq 0$ in general.

Solution: Baker–Campbell–Hausdorff formula: (Dynkin).

$$\log(e^X \cdot e^Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1 \le i \le n}} \frac{\left(\sum_{i=1}^n (r_i+s_i)\right)^{-1}}{r_1!\, s_1! \ldots r_n!\, s_n!} \underbrace{[X^{r_1} Y^{s_1} \ldots X^{r_n} Y^{s_n}]}_{\text{multicommutator}}$$

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) - \frac{1}{24}[Y,[X,[X,Y]]] - \ldots$$

$$= X + \frac{ad_X}{1-e^{-ad_X}}\, Y + O(Y^2),$$

in which $\frac{ad_X}{1-e^{-ad_X}} = Y + \frac{1}{2}[X,Y] + \sum_{k=2}^{\infty} \frac{B_k}{k!}\, ad_X^{\,k}(Y).$

<u>Remark</u> $\log(e^x e^y)$ is associative in the sense that

$\log(e^x e^y) = ch(x,y)$

$$ch(x, ch(y,z)) = ch(ch(x,y),z).$$

<u>Conjecture</u> (Kashiwara–Vergne) (1978)

There exist $A(x,y)$ and $B(x,y)$ Lie series in x and y such that:

1. $x + y - \log(e^x e^y) = (1-e^{-ad_x})\cdot A + (e^{ad_y}-1)\cdot B$

2. $tr_g(ad_x \circ \partial_x A + ad_y \circ \partial_y B) = \frac{1}{2} tr_g\left(\frac{ad_x}{e^{ad_x}-1} + \frac{ad_y}{e^{ad_y}-1} - \frac{ad_z}{e^{ad_z}-1}\right)$

in which
- $z = \log(e^x e^y)$
- $\partial_x A : g \to g$

  $\partial_x A(u) = \frac{d}{dt} A(x+tu, y)\Big|_{t=0}$

<u>Theorem</u> (K.V.)

K.V. conjecture $\Rightarrow$ Duflo isomorfism.

<u>Theorem</u> (Torossian) If g is quadric, then

KV1 $\Rightarrow$ KV2.

g Quadric, if there exist non-degenerate symmetric bilinear form $Q: g \times g \to \mathbb{K}$ with

$$Q([x,y],z) + Q(y,[x,z]) = 0.$$

For example g - semi-simple, Q - is killing.

<u>Theorem</u> KV conjectures holds true for all finite-dimensional Lie algebra.

<u>Proofs</u>. Meinrenken (2006) $\longrightarrow$ Kontsevich grafical calculus

Torossian (2008) $\longrightarrow$ Drinfeld associators.