Hall algebras: the first touch.

⊙ $\mathcal{A}$ - abelian category (with "nice" properties).

$\mathcal{A} = Rep_{\mathbb{F}_q} Q$

$\mathcal{A} = Coh(X)$
X - projective scheme

$\mathbb{H}_{\mathcal{A}}$ - Hall algebra

•) as vector space it is generated by iso classes of the objects in $\mathcal{A}$.

•) multiplication of X and Y encodes the extensions of X by Y...

Theorem (Ringel) Let Q be a Dynkin quiver and let $\mathfrak{g}$ be a Lie algebra associated to Q. There exist a unique isomorphism

$$\mathcal{U}_q^+(\mathfrak{g}) \simeq \mathbb{H}_{Rep_{\mathbb{F}_q} Q}.$$

Theorem (Kapranov...)

$\mathcal{L}$ = Loop algebra $\quad \mathcal{U}_q^+(\mathcal{L}) \simeq \mathbb{H}_{Coh\,X}$

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Hall algebras: the first touch

① ~~B~~ Basic notions:

Let A be an abelian category.

a) $A = \mathrm{Rep}\,Q$ – representations of some quiver

b) $A = \mathrm{Coh}(X)$ – coherent sheaves over a projective scheme X.

By $\mathrm{Ext}^i(X,Y)$ – i-th extensions $\mathbb{Z}$-module, $i=0,\ldots$

$i=0$, $\mathrm{Ext}^0(X,Y) = \mathrm{Hom}(X,Y)$

$$\mathrm{Ext}^i(X,Y) \simeq \{p : 0 \to Y \to \ldots \to X \to 0\}/\sim$$

A – is hereditary if $\mathrm{Ext}^i(X,Y)=0$, for any $X,Y \in \mathrm{Ob}(A)$ and $i>1$.

A – is finitary if

a) $\#\,\mathrm{Hom}(X,Y) < \infty$, $X,Y \in \mathrm{Ob}(A)$

b) $\#\,\mathrm{Ext}^1(X,Y) < \infty$, $X,Y \in \mathrm{Ob}(A)$.

~~A~~ Let k – be any field. Then A is k-linear if $\mathrm{Ext}^i(X,Y)$ – are k-vector spaces for all i

In most examples A is $\mathbb{F}_q$-linear, and

$$\dim \mathrm{Hom}(X,Y) < \infty, \quad \dim \mathrm{Ext}^1(X,Y) < \infty.$$

So, A still be finitary.

Grothendieck ~~gr~~ group $\boxed{K(A)}$ of A is a group generated by the classes of $[A]$ of objects subject to relations $[A]+[B]=[A+B]$ if $\exists$ a sequence

$$0 \to A \to A+B \to B \to 0 \text{ with}$$

$A \in [A]$, ...

Euler form

Assume that A is finitary category and that a global dimension $\mathrm{gldim}\, A < \infty$.

$$\min_i \{ \mathrm{Ext}^{i+1}(X,Y) = 0 \mid \forall X,Y \in \mathrm{Ob}(A) \}$$

$X, Y \in \mathrm{Ob}(A)$

$$\langle X,Y \rangle := \left(\prod_{i=0}^{\infty} \# \mathrm{Ext}^i(X,Y)^{(-1)^i} \right)^{1/2}$$

Since $\mathrm{gldim}\, A < \infty$, $\langle X,Y \rangle$ is well defined.
Also it is not hard to show that it depends only on the classes X, Y in $K(A)$ hence.

$$\langle \cdot,\cdot \rangle : K(A) \times K(A) \to \mathbb{C}.$$

Multiplicative Euler form (symm.) is defined to be

$$(X,Y) := \langle X,Y \rangle \cdot \langle Y,X \rangle.$$

when A is k-linear then usually people takes

$$\langle M,N \rangle = \sum_i (-1)^i \dim \mathrm{Ext}^i(M,N) \text{ and}$$

$$(M,N) = \langle M,N \rangle + \langle N,M \rangle.$$

② <u>Examples</u>

$\mathcal{A} = \mathrm{Vect}$ - category of k-vector spaces.

what is a global dimension of $\mathcal{A}$ - ?

$F_2 \to (F_2)^2$

$\langle V, W \rangle_m = \# \mathrm{Hom}_{\mathbb{F}_q}(V, W) =$

$= \cancel{|\mathbb{F}_q| \cdot \dim V \cdot \dim W}\ |\mathbb{F}_q|^{\cancel{\dim V + \dim W}}\ \sqrt{|\mathbb{F}_q|}^{\dim V \cdot \dim W}$

$\langle V, W \rangle_m = |\mathbb{F}_q| \cdot \dim V \cdot \dim W.$

$(N, M)_a := \langle N, M \rangle_a + \langle M, N \rangle_a$

$\langle N, M \rangle_a := \sum_{i=1}^{\infty} (-1)^i \dim \mathrm{Ext}^i (N, M).$

~~Q~~ Q - be finite quiver

Q - can have multiple edges and cycles but no loops

$Q = (Q_0, Q_1)$ $\quad \beta \in Q_1,\ h(\beta) \in Q_0$ - head

vertices edges $\quad t(\beta) \in Q_0$ - tail.

A <u>representation</u> of Q is a ~~k~~ collection of k-vector

~~spaces~~ $X = ((X_i)_{i \in Q_0}, X_{\beta: i \to j}: X_i \to X_j)$,

vector spaces maps.

$\mathrm{Hom}(X, Y) = (f_i)_{i \in Q_0} \quad f_i : X_i \to Y_i$

$$\begin{array}{ccc} X_i & \xrightarrow{X_\beta} & X_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ Y_i & \xrightarrow{Y_\beta} & Y_j \end{array}$$ for every $\beta \in Q_1$.

Let $\operatorname{Rep} Q$ be a category of representations of Q over $\mathbb{F}_q$

$\operatorname{Rep} Q$ – is abelian, k-linear, hereditary.

What is the simplest quiver?
if Q is just a point $\Rightarrow \operatorname{Rep} Q \simeq \operatorname{Vect}$

Let $X \in \operatorname{Rep} Q$, $(\dim X_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$

$\boxed{\text{Home work}}$ Show that a map $\underline{\dim} : K(\operatorname{Rep} Q) \to \mathbb{Z}^{|Q_0|}$

$\underline{\dim} X \mapsto (\dim X_i)_{i \in Q_0}$ – is an isomorphism of groups

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$\operatorname{Ext}^i(X, Y) = 0$ if $i > 1$.

$\langle X, Y \rangle_a = \dim \operatorname{Hom}(X, Y) - \dim \operatorname{Ext}(X, Y)$.

$q(X) := \langle X, X \rangle$ – Tits Form.

One checks that:

$$\langle X, Y \rangle_a = \sum_{i \in Q_0} \dim X_i \cdot \dim Y_i - \sum_{\beta : i \to j} \dim X_i \cdot \dim Y_j$$

Simple objects in $\operatorname{Rep} Q$ are formed by

$E_i \in \operatorname{Rep} Q$. $(E_i)_j = \begin{cases} k, & j = i \\ 0, & j \neq i \end{cases}$ $E_\beta = 0$, $\beta \in Q_1$

Let c_{ij} – be a number of ~~oriented~~ arrows between i and j

<u>Home work</u> Show that $\langle S_i, S_j \rangle_a = \delta_{ij} - c_{ij}$

$\langle S_i, S_j \rangle_m = q^{\frac{1}{2}}(\delta_{ij} - c_{ij})$

~~the matrix of of form~~

<u>theorem</u> Let K - be an algebraically closed field. Then for any algebra A of global dimension 1. There exist a quiver Q such that Mod-A is equivalent to <u>Rep Q</u>

<u>Invariants</u>: $K(\mathrm{Rep} A)$

$\langle\ \rangle_{a,m}$ - Euler forms

<u>indecomposable</u> <u>objects</u> - ?

Rep Q - is Krull-Schmidt $\Leftrightarrow$ every $X = \bigoplus N_i$, N_i - indecomposable.

Q - is finite type - if Rep Q has a finite number of indecomposables

Q - tame - if $\exists$ finitely-many one parametric idem.

Q - wild - elsewhere.

<u>Theorem</u> (Gabriel) Q has finite type $\Leftrightarrow$ Δ_Q - is positive defined (i.e. $\mathfrak{g}$ is a ~~finite~~ simple Lie algebra). $\exists$ bijection between indecomposable and <u>positive</u> <u>roots</u> Δ^+.

(3) Hall algebra of a category

Let $\mathcal{A}$ be a finitary cat.

$\mathcal{X} = Ob(\mathcal{A})/\sim$ – set of isomorphism classes of objects in $\mathcal{A}$.

$$H_{\mathcal{A}} = \bigoplus_{M \in \mathcal{X}} \mathbb{C}[M].$$

X, Y, Z in $Ob(\mathcal{A})$,

$$\mathcal{P}^{Z}_{X,Y} = \{0 \to X \to Z \to Y \to 0\}$$

$P^{Z}_{X,Y} = |\mathcal{P}^{Z}_{X,Y}|$, Hom$(X,Z)$, Hom$(Z,Y)$ – are finite.

$a_X = |\mathrm{Aut}(X)|$

Proposition (Ringel) the following defines on $H_{\mathcal{A}}$ the structure of associative algebra

$$[X]\cdot[Y] = \langle X, Y\rangle_m \sum_{R} \frac{1}{a_X a_Y} P^{R}_{X,Y}$$

the unit. $i: \mathbb{C} \to H_{\mathcal{A}}$ is given by $i(c) = c[0]$ 0 is zero obj in $\mathcal{A}$

$$\mathbb{H}_{\mathcal{A}} = \{ f : \mathcal{X} \to \mathbb{C} \mid \mathrm{Supp}(f) < \infty \}.$$

$[M] \mapsto 1_M$ – characteristic function of M.

$$(f \cdot g)(R) = \sum_{Q \subseteq R} \langle R/Q, Q \rangle_m f(R/Q) g(Q).$$

<u>Homework</u> prove Ringel's proposition.

~~Fac~~

<u>Observation</u> $\mathbb{H}_A$ encodes short exact sequences in A.

$\mathbb{H}$ - graded by $K(A)$.

$$\mathbb{H}_A = \bigoplus_{\alpha \in K(A)} \mathbb{H}_A[\alpha], \quad \mathbb{H}_A[\alpha] = \bigoplus_{[M] \in \alpha} \mathbb{C}[M]$$

$\mathcal{A}$ - semisim

$$[S_i] \cdot [S_j] = [S_i \oplus S_j] = [S_j] \cdot [S_i]$$

$$[S_i] \cdot [S_i] = |\mathrm{End}(S_i)|^{1/2} (|\mathrm{End}(S_i)| + 1) [S_i \oplus S_i].$$

a_{44}

$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$ $\quad \mathrm{Hom}(S_1, S_2) = 0$

$\mathrm{Hom}(S_2, S_1) = 0.$

$\mathrm{Ext}^1(S_1, S_2) = \{0\}.$

$\mathrm{Ext}^1(S_2, S_1) = K.$

$\langle S_1, S_2 \rangle_m = q^{-\frac{1}{2}}$ $\quad I_{12}$

$\mathrm{Hom}(S_2, I_{12}) = k$

$[S_1][S_2] = v^{-1}([S_1 \oplus S_2] + [I_{12}]).$

$[S_2][S_1] = [S_1 \oplus S_2]$

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<u>Let</u> $k = \mathbb{F}_q$, $\mathcal{A} = \mathrm{Vect}_{\mathbb{F}_q}$.

$S = k$, $X \simeq k^m$

$\mathrm{Hom}(k, k) \simeq k$, $\mathrm{Ext}(k, k) = \{0\}$, $\langle S, S \rangle_m = q^{\frac{1}{2}}$

$$[k^m] \cdot [k^n] = \langle k^m, k^n \rangle_m \frac{P^{k^{m+n}}_{k^n k^m}}{a_{k^m} a_{k^n}} [k^{m+n}]$$

$$= \# Gr(n, m+n) = \binom{m+n}{n}_q$$

$[n]_q! = [n-1]_q \ldots [1]_q$, $\quad [n]_q = \frac{q^n - 1}{q - 1}.$