Formality theorem

① Formal deformations

Let k - be a field.

A - an associative algebra over k.

$k[[t]]$ - ring of formal power series over k.

$A[[t]]$ - $k[[t]]$-module of formal power series

$$\sum_{n=0}^{\infty} a_n t^n \text{ with } a_n \in A.$$

Def. A <u>formal</u> <u>deformation</u> of A is an associative $k[[t]]$ bilinear map

$$m : A[[t]] \times A[[t]] \longrightarrow A[[t]]$$

Such that for $a, b \in A$ we have

$$m(a,b) = ab + \sum_{n\geq 1}^{\infty} m_i(a,b) \cdot t^i. \qquad (*)$$

in which ab is a usual product on A.

Rem • Sometimes m is denoted by $*$, and is called $*$-product.

• $(*)$ means that

$$u * v \equiv uv \mod t \cdot A[[t]]$$

for all formal power series $u, v \in A[[t]]$.

• Associativity of m can by formal expressed as

$$m(m(a,b),c) - m(a, m(b,c)) = 0$$

for all $a,b,c \in A$.

<u>Def. 2</u> An associative algebra A is called <u>Poisson algebra</u> if it comes along with a bracket
$$\{\cdot,\cdot\} : A \times A \to A \text{ which satisfies}$$
the following rules

(1) $\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0$

for all $a,b,c \in A$, and

(2) $\{a, bc\} = \{a,b\}c + b\{a,c\}$

($\rightsquigarrow \{\cdot,\cdot\}$ acts as derivation on product in A).

<u>Example</u> Any associative algebra A with bracket
$$\{a,b\} = ab - ba.$$

These two structures are naturally linked.

$$\boxed{\text{Formal deformation of } A} \xrightarrow{\text{gives rise}} \boxed{\text{Poisson structure on } A}$$

<u>Lemma</u> Given formal deformation $m : A[[t]] \times A[[t]] \to A[[t]]$ its first order part gives rise to a Poisson bracket as

$$\{a,b\} := \frac{1}{2}\left(m_1(a,b) - m_1(b,a)\right)$$

Proof. $m(a,b)$ is associative. Hence, in particular

$$m_1(a, m_1(a,b)) = m_1(m_1(a,b), c) \quad \forall a,b,c \in A.$$

and then Jacobi identity follows.

(2) is also easy to verify. $\square$.

Deformation Quantization Problem:

Given a Poisson algebra A find formal deformation $m : A[[t]] \times A[[t]] \to A[[t]]$, such that $\{a,b\} = \frac{1}{2}\left(m_1(a,b) - m_1(b,a)\right)$.

Crucial point: m should be associative.

Theorem (Kontsevich, 97) If $A = C^\infty(M)$ is the algebra of smooth functions on differentiable manifold M

$\Rightarrow$ each Poisson bracket lifts to associative formal deformation.

② Examples:

a) Consider $M = \mathbb{R}^2$ with Poisson bracket

$$\{f,g\} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\cdot\frac{\partial g}{\partial x_1}$$

$\Rightarrow$ Kontsevich's construction yields to formal deformation of $C^\infty(\mathbb{R}^2)$.

$$f * g = f\cdot g + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} t^1 + \dots + \frac{\partial^n f}{\partial x_1^n}\cdot\frac{\partial^n g}{\partial x_2^n}\frac{t^n}{n!} + \dots$$

b) Consider g - finite-dimensional real Lie algebra g^* - its dual, $A = C^\infty(g^*)$ - smooth functions of g^*.

It is just standart that given $f \in A$ and $x \in g^*$ then $(df)_x$ is an element of g. Using this identification we define a Poisson bracket on A by

$$\{f,g\}(x) = x\left([(df)_x, (dg)_x]\right), \quad x \in g^*.$$

Kontsevich's construction gives a canonical associative product $*$ on $A[[t]]$, which is linked with $U(g)$.

Let $S(g)$ be symmetric algebra on g. Then $S(g)$ are polinomial functions on g^* via the pairing $g^* \times S(g) \to \mathbb{R}$. Moreover $B = S(g)[t]$ is ~~$*$~~-subalgebra of $A[[t]]$.

Moreover the inclusion $g \to B$ induces an isomorfism

$$U_{hom}(g) \xrightarrow{\sim} S(g)[t]$$

where $U_{hom}(g)$ – homogeneous enveloping algebra, i.e. $\mathbb{R}[t]$-algebra generated by g with rel.

$$XY - YX - t[X, Y], \quad X, Y \in g$$

$$\Rightarrow S(g)[t]/(1-t)B = U_{hom}(g)/(1-t)U_{hom}(g) \simeq U(g).$$

Example (Mathieu)

Given Poisson structure on A $\Leftarrow$ does not com $\{$ Formal deformations of A

"Outlines" Take g – finite-dim real Lie algebra $g \otimes_{\mathbb{R}} \mathbb{C}$ – simple and not isomorfic to $sl_2(\mathbb{C})$.

Bracket of g extends to Poisson Bracket on $S(g)$

$A = S(g)/g^2$, g^2 – all monomials in $S(g)$ generated by mon. of degree 2. – which is Poisson ideal!

Hence A is Poisson algebra.

By some consideration Mathieu showed that $\{\cdot,\cdot\}$ cannot come from formal deformation.

(3) Kontsevich's "explicit" formula

Let $d \geq 1$, M – non-empty open subset of $\mathbb{R}^d$.

$A = C^\infty(M)$ – smooth functions,

$\{\cdot,\cdot\}$ – Poisson bracket on A.

Lemma There are unique smooth functions α^{ij}, $1 \leq i < j \leq d$ such that

$$\{\cdot,\cdot\} = \sum_{i<j} \alpha^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

i.e. for any $f, g \in C^\infty(A)$

$$\{f,g\} = \sum_{i<j} \alpha^{ij} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right).$$

We find formal deformation in the form

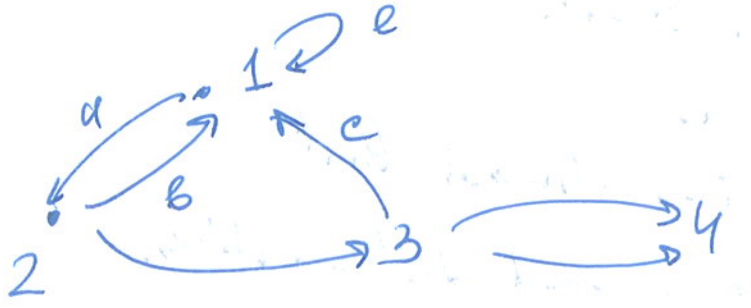
$$f * g = f \cdot g + \sum_{i=1}^{\infty} B_i(f,g)\, t^i$$

Find $B_i(f,g)$ – as combinations of partial derivatives of f, g & α^{ij}.

For this we need some combinatorics.

A Quiver $\Gamma = (\Gamma_0, \Gamma_1)$, $s, t : \Gamma_1 \to \Gamma_1$,

vertices (Γ_0), arrows (Γ_1), associate for any arrow a its source and target



$s(a) = 1$, $s(e) = t(e) = 1$
$t(a) = 2$,

Let $n \geq 0$, We define G_n – to be the set of quivers Γ such that:

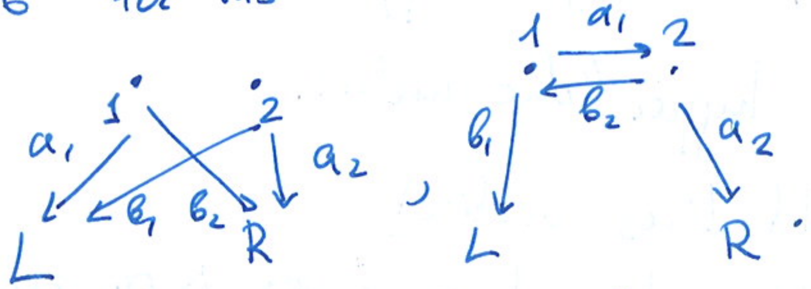
- $\Gamma_0 = \{1, \dots, n\} \cup \{L, R\}$, L, R two symbols.
- $\Gamma_1 = \{a_1, b_1, \dots, a_n, b_n\}$ a_i, b_i – symbols.
- for each i, $s(a_i) = s(b_i) = i$
- Γ has no loops, has no double arrows.

Examples

$$G_0 = \left\{ \overset{\cdot}{L} \quad \overset{\cdot}{R} \right\}.$$

$$G_1 = \left\{ L \xleftarrow{a_1} \underset{1}{\cdot} \xrightarrow{b_1} R \right\} \cup \left\{ L \xleftarrow{b_1} \underset{1}{\cdot} \xrightarrow{a_1} R \right\}.$$

$|G_2| = 36$ For instanse



$|G_n| = (n(n+1))^n$

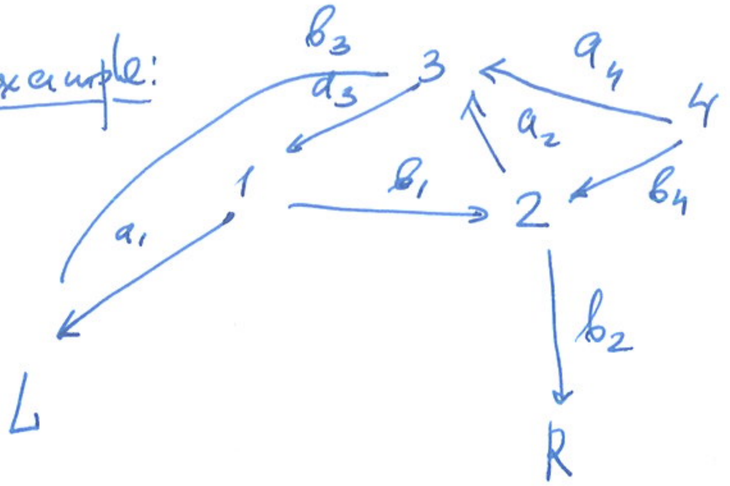
Given $\Gamma \in G_n$ we ~~have~~ define

$$\mathrm{Crazy}_{\Gamma,\alpha}(f,g) = \sum \left(\prod_{i=1}^{n} \left(\prod_{a \in \Gamma(?,i)} \partial_{I(a)} \right) \alpha^{I(a_i), I(b_i)} \right) \cdot \left(\prod_{a \in \Gamma(?,L)} \partial_{I(a)} \right)(f) \cdot \left(\prod_{a \in \Gamma(?,R)} \partial_{I(a)} \right)(g).$$

Sum ranges over all maps $I : \Gamma_1 \to \{1, \dots, d\}$

$\Gamma(?, v)$ – arrows with target v.

For example:



In this case

Crazy $\cancel{\mathcal{B}}_{\Gamma,\alpha}(f,g) = \sum (\partial_{i_3}\alpha^{i_1,j_1})(\partial_{j_1}\partial_{j_4}\alpha^{i_2,j_2})(\partial_{i_2}\partial_{i_4}\alpha^{i_3,j_3})$

$$(\partial_{i_1}\partial_{j_3} f)(\partial_{j_2} g).$$

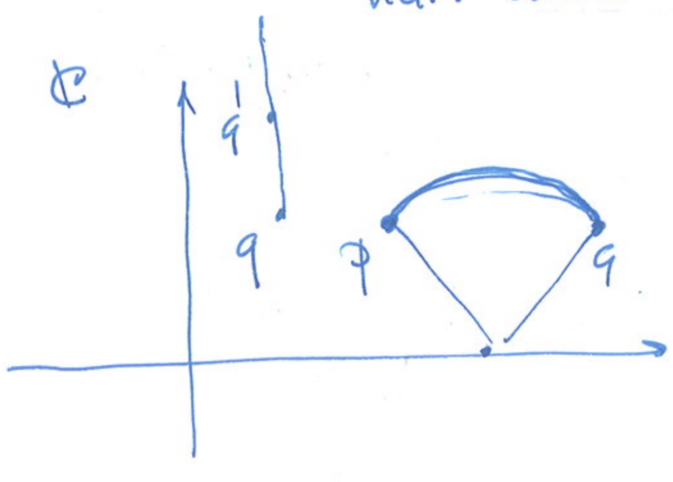
We define $\quad B_n(f,g) = \sum\limits_{\Gamma \in G_n} w_\Gamma \cdot B_{\Gamma,\alpha}$

For some w_Γ which we gonna define.

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ — upper half plane.

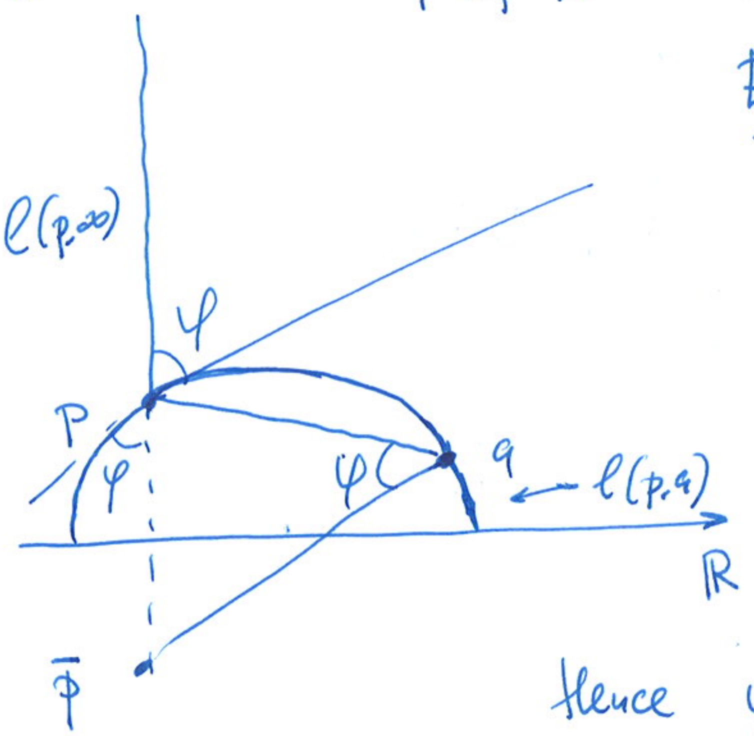
Endow $\mathcal{H}$ with hyperbolic metric.

Geodesics: half-lines vertical,
half-circle whose center is on real axis



For two distinct points $p, q \in \mathcal{H}$, let $\ell(p,q)$ be geodesic from p to q, $\ell(p,\infty)$ half-line from p to infinity.

We denote $\varphi(p,q)$ the angle between $\ell(p,\infty)$, $\ell(p,q)$.



<u>Exercice</u> show that

$$\varphi(p,q) = \arg\left(\frac{q-p}{q-\bar{p}}\right) = \frac{1}{2i}\log\left(\frac{q-p}{q-\bar{p}}\cdot\frac{\bar{q}-p}{\bar{q}-\bar{p}}\right).$$

Hence $\varphi(p,q)$ is analytic

Given $n \geq 0$, let $\mathcal{H}_n$ be the set of n-tuples $(p_1,\dots,p_n)$ of distinct points of $\mathcal{H}$.

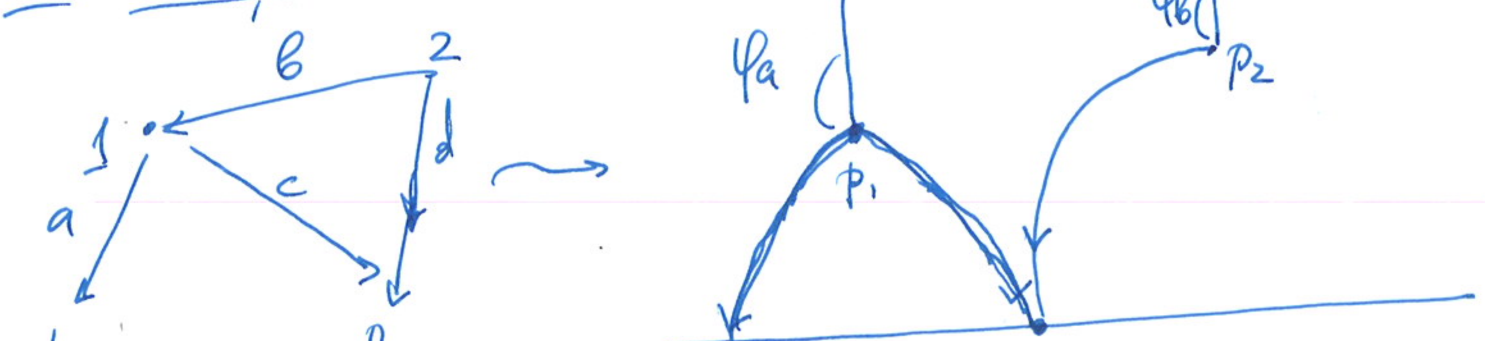
"Geometrically" $\mathcal{H}_n$ is the set of "geodesic drawings":

vertices $1,\dots,n$ of $\Gamma \in G_n \longmapsto$ points p_i

vertices $L, R \longmapsto 0, 1$ on real axis

arrows $i \to j \longmapsto$ geodesic segment from p_i to p_j.

<u>For example</u>



Define a function $\varphi_a(p_1,\dots,p_n) = \varphi(p_{s(a)}, p_{t(a)})$, $\varphi_a : \mathcal{H}_n \to \mathbb{R}$

for each arrow a of Γ.

Finally

$$\omega_\Gamma := \frac{1}{(2\pi)^n} \int_{\mathcal{H}_n} \bigwedge_{i=1}^{n} d\varphi_{a_i} \wedge d\varphi_{b_i}$$

<u>Lemma</u> Integral converges absolutely.

<u>Teorem</u> (Kontsevich)

$$f * g = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\Gamma \in G_n} \omega_\Gamma \cdot \mathrm{Crazy}_{\Gamma,\alpha}(f,g).$$

defines a formal quantization of poisson bracket on $A = C^\infty(M)$.