Lecture 2 "Algebras $\rightleftarrows$ quivers"

Plan:

1. Algebras, their radicals. Representations and pointed algebras.
2. Quivers and their representations.
3. Algebra $\rightleftarrows$ quiver.

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1). Let $k = \bar{k}$ be algebraically closed field.
A – fin. dim alg over k (vector space + ass. product with 1).

Examples 1. $A = k$
2. $A = \mathbb{C}[x_1, \dots, x_n]$
3. $A = k[x]/(x^2)$ – algebra of dual numbers
$$\{a + bx \mid a, b \in k,\ x^2 = 0\}$$
4. $A = M_n(k)$, $A = U_n(k)$. 5. $A = k\langle x_1, x_2 \rangle$ ...

(Jacobson) Radical of $A \overset{\text{def}}{=} \bigcap$ maximal right ideals in A
$= \bigcap$ maximal left ideals in A

$\Rightarrow J(A)$ is two-sided ideal.

A is called semi-simple if $J(A) = 0$

Example $A = M_n(k)$ is semi-simple as $J(A) = 0$
$A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, $J(A) = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$

$A = \bigoplus M_n(k)$ is semi-simple.

Exercices

1) Let G be finite group, $k[G]$ its group algebra. Show that $k[G]$ is semi-simple.

2) Let $f: A \to B$ – homomorfism show that $f(J(A)) \subseteq f(J(B))$.

Teorem (Wedderburn–Artin).

$$A/J(A) \cong M_{d_1}(k) \oplus \dots \oplus M_{d_n}(k). \quad (*)$$

So any semi-simple algebra has a form as in Ex. 1e).

We call A basic (pointed) if all d_i in $(*)$ are 1.

that is

$$A \text{ is basic} \iff A/J(A) \cong \prod_{i=1}^{n} k.$$

Examples $A = M_n(k)$ is basic $\iff n = 1$.

- Let $A = k[x]/x^2$ – dual numbers.
  $J(A) = \langle x \rangle$ – is one dimensional.
  therefore $A/J(A) = k$ – A is pointed.
- Same, if $A = k[x]/x^m$
  $J(A) = \langle x \rangle$, – $(m-1)$-dim.
  and $A/J(A) = k \Rightarrow$ A is pointed
- $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, $J(A) = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$

  $A/J(A) \cong \bigoplus_{i=1}^{2} k$

  $\Rightarrow$ A is pointed

- $A = \begin{pmatrix} k & k[x]/x^2 \\ 0 & k[x]/x^2 \end{pmatrix}$ — algebra, with

$$J(A) = \begin{pmatrix} 0 & k[x]/x^2 \\ 0 & x\,k[x]/x^2 \end{pmatrix}$$

Therefore we have $A/J(A) \cong \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \bigoplus_{i=1}^{2} k$

— pointed

<u>Representation</u>:

A <u>representation</u> of A is a Vector space V with algebra homomorfism $\rho: A \to \mathrm{End}(V)$

<u>Ex.</u>
- $V=0$, $\rho=0$.
- $V=A$, $\rho: A \to \mathrm{End}(A)$ — regular Representation
  $a \mapsto (b \mapsto ab)$
- Seja $A=k$, any rep of A is a vector space in which A acts by multiplication by scalars.

Given two representations (V_1, ρ_1) and (V_2, ρ_2) of A, a <u>morfism</u> is a linear map $\varphi: V_1 \to V_2$ such that

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(a)"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & V_1 \arrow[d, "\varphi"] \\ V_2 \arrow[r, "\rho_2(a)"] & V_1 \end{tikzcd}$$

for any $a \in A$, that is $\varphi\rho_1(a) = \rho_2(a)\varphi$

Therefore we can form a category
Rep A, in which <u>object</u> — fini-dim reps of A
<u>morfism</u> — morfisms between them

<u>Pointed</u> algebras play a crutial role due to the following theorem:

<u>Theorem</u> For any finite-dim algebra A, there exists a <u>basic</u> (<u>pointed</u>) algebra B such that <u>Rep A is equivalent to Rep B.</u> (Morita equivalence)

<u>Obs</u> that is "it is enough" to study just representations of basic algebras.

(2.) <u>Quivers and their representations</u>

Quiver Q is an (finite) <u>oriented graph</u>.

We denote $Q = (Q_0, Q_1)$

<u>vertex</u> (Q_0) <u>arrows</u> (Q_1)

A <u>representation</u> of Q is an assignment to any
<u>vertex</u> $i \in Q_0$ — a vector space V_i over k.
<u>arrow</u> $\alpha : i \to j \in Q_1$ — a linear map $V_\alpha : V_i \to V_j$

So let Rep(Q) be a category of finite-dim ~~modules~~ representations of Q. (5

$$Q \longmapsto kQ \quad - \quad$$ **Basis**: oriented path in Q; **multiplication**: concatenation of path in Q.

kQ — path algebra

Example $Q: \; 1 \xrightarrow{\alpha} 2$

Then, kQ has a basis with 3 elements:

p_1, p_2 — trivial paths in vertices

α — path of length 1.

Obviously:

	p_1	p_2	α
p_1	p_1	0	α
p_2	0	p_2	0
α	0	α	0

Easy to see that $p_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $p_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

gives an isomorfism $kQ \simeq \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$

Exercices Show that $J(kQ)$ is generated by all arrows in Q. (If Q is finite acyclic).

Using this exercise we have that

$$kQ / J(kQ) \simeq \prod_{i \in Q_0} k \Rightarrow$$ kQ is pointed algebra

Obs

kQ is finite-dim $\Leftrightarrow$ Q has no oriented cycles.

<u>Theorem</u> Let Q be finite acyclic quiver then there is an isomorfisms

$$\operatorname{Rep} Q \simeq \operatorname{Rep}(kQ)$$

<u>Obs</u> Therefore "studing representation of kQ" is the same as "studying repr of Q"

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③ <u>Gabriel</u> <u>Quiver</u>

We saw that having finite acyclic quivel Q we get

finite-dimensional algebra kQ (which is pointed).

Now given basic algebra A we will construct a finite quiver Q_A (called Gabriel quiver).

<u>Recall</u>: Let A be a finite-dim algebra.

$e \in A$ is <u>idempotent</u> if $e^2 = e$.

two idempotents $e, f \in A$ are <u>ortogonal</u> if $ef = 0 = fe$

Idempotent $e \in A$ is called <u>primitive</u> if it is imposible to write $e = e_1 + e_2$, in which e_1, e_2 two non-zero idempotents.

So now let A is basic finite-dim algebra and $\{e_1, \dots, e_n\}$ — full set of primitive ortogonal idempotents of A (i.e. $e_1 + \dots + e_n = 1$).

Gabriel Quiver $Q_A = (Q_0, Q_1)$ is defined as follows:

<u>vertex</u>: element of the set $\{e_1, \dots, e_n\}$.

number of <u>arrows</u> between e_i and e_j $\;=\;$ $\dim e_i \, {}^{J(A)}/_{J^2(A)} e_j$

<u>Examples</u>
- Let $A = k[x]/x^n$. $e = 1$ is unique non-zero idempotent of A. $\Rightarrow$ Q_A has one vertex.
  We saw that $J(A) = (x)$ (generated by x)
  $\Rightarrow J^2(A) = (x^2)$.
  Therefore $e \dim {}^{J(A)}/_{J^2(A)} e = 1$, and

$$Q_A: \quad \begin{tikzcd} 1 \arrow[loop above, "\alpha"] \end{tikzcd}$$

- $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$. Let fix the following complete set of prim idemp. $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
  $\Rightarrow$ Q_A has 2 vertex.
  $J(A) \simeq \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$, $J^2(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\Rightarrow$ ${}^{J(A)}/_{J^2(A)} \cong k$.
  $e_i \, {}^{J(A)}/_{J^2(A)} e_i = 0$, $e_1 \, {}^{J(A)}/_{J^2(A)} e_2 \cong \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$, $e_2 \, {}^{J(A)}/_{J^2(A)} e_1 = ($

Therefore in this case

$$Q_A : \quad 1 \xrightarrow{\alpha} 2$$

• let $A = \begin{pmatrix} k & k[x]/x^2 \\ 0 & k[x]/x^2 \end{pmatrix}$ we have

$$J(A) = \begin{pmatrix} 0 & k[x]/x^2 \\ 0 & x\,k[x]/x^2 \end{pmatrix}, \quad J^2(A) = \begin{pmatrix} 0 & x\,k[x]/x^2 \\ 0 & 0 \end{pmatrix}$$

Therefore $J(A)/J^2(A)$ has dimension 2.

and Q_A has two arrows.

A has complete set of primitive idemp. $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Easy to see that

$$\dim e_i \, J(A)/J^2(A) \, e_j = \begin{cases} 1, & (i,j) = (1,2), \text{ or } (i,j) = (2,2) \\ 0, & \text{otherwise.} \end{cases}$$

hence Q_A:

$$\begin{tikzcd} 1 \arrow[r, "\beta"'] & 2 \arrow[loop above, "\alpha"] \end{tikzcd}$$

The following theorem holds.

<u>Theorem</u> let A - basic, conected, fini-dimensional algebra. There exist an admissible ideal in kQ_A such that

$$A \simeq kQ_A/I.$$

$I \triangleleft kQ_A$ is called admissible if $R_Q^m \subseteq I \subseteq R_Q^2$, in which

R_Q - two sided ideal in kQ generated by arrows in Q.

So given algebra A, there exist basic algebra B such that

$$\operatorname{Rep} A \underset{\substack{\uparrow \\ \text{equivalence}}}{\simeq} \operatorname{Rep} B \underset{\substack{\uparrow \\ \text{isomorfism}}}{\simeq} \operatorname{Rep} kQ_B/I$$