Hall algebras: Quantum Kac-Moody algebras $U_q(\mathfrak{g})$

⓪ $\boxed{\text{Why?}}$

Theorem (Ringel) $\quad U^+_{\sqrt{q}}(\mathfrak{g}) \simeq \mathbb{H}_{\mathrm{Rep}_{\mathbb{F}_q} Q}$

- • $\mathbb{H}_A$ – for an arbitrary "nice" category A
- •• $\mathrm{Rep}_{\mathbb{F}_q} Q$ – representations of Quiver
- ••• $U_v(\mathfrak{g})$ – aim for today.

(the first two: we know.)

$\boxed{\text{Picture}}$

$$\begin{tikzcd}
\text{Simple Lie algebras } \mathfrak{g} \arrow[r, shift left, "\text{Serre + Cartan}"] \arrow[d, phantom, "\cup"] & \text{generators + relations + positive def Cartan matrices} \arrow[l, shift left] \arrow[d, "\text{Kac Moody}", "\cancel{\text{positive}}"'] \\
\text{Kac-Moody algebras } \mathfrak{g} \arrow[d, "\text{PBW theorem}"', "\text{Functor}"] & \text{generators + relations, Cartan matrices} \arrow[l] \\
\text{Univ. enveloping algebra } U(\mathfrak{g}) \arrow[r, shift left, "\text{Deformation } q"] & \text{Quantum group } U_q(\mathfrak{g}) \text{ (Drinfeld, Jimbo)} \arrow[l, shift left, dashed, "q=1"]
\end{tikzcd}$$

① Simple Lie algebras

A complex Lie algebra $\mathfrak{g}$ is a $\mathbb{C}$-vector space with a bracket $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ with some "standart" conditions.

A Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ has no non-trivial ideals and $\mathfrak{g}$ is non-abelian.

Given a simple Lie algebra $\longrightarrow$ "Cartan matrices"

$$(a_{ij})_{i,j=1}^{n}, \quad a_{ij}\in\mathbb{Z},$$

$$(*) \quad a_{ii}=2, \; a_{ij}\le 0, \text{ if } i\neq j$$

$$a_{ij}=0 \Rightarrow a_{ji}=0.$$

Prop (Cartan) Matrix with $(*)$ is a Cartan of some simple Lie algebra $\Longleftrightarrow A$ is positive def.

Theorem (Serre) Each simple Lie algebra is isomorphic to the Lie algebra generated by the elements $\{e_i, h_i, f_i \mid i=1\ldots n\}$ subject to

$$[h_i,h_j]=0$$

$$[h_i,e_j]=a_{ji}e_j \qquad \forall i,j$$

$$[h_i,f_j]=-a_{ji}f_j\, ,$$

$$[e_i,f_j]=\delta_{ij}h_i$$

Serre relations
$$\begin{cases} \operatorname{ad}^{1-a_{ij}}(e_i)(e_j)=0 \\ \operatorname{ad}^{1-a_{ij}}(f_i)(f_j)=0, \end{cases}$$

(a_{ij}) – Cartan matrix of $\mathfrak{g}$, and $\operatorname{ad}(x)(y)=[x,y]$.

<u>Ex</u> Let $A=(2)$, we have 3 generators h, f, e.

$$\left.\begin{aligned}&[h,e]=2e\\&[h,f]=-2f\\&[e,f]=h\end{aligned}\right\} \longrightarrow sl_2$$

Sending $h \mapsto \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $e \mapsto \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f \mapsto \begin{pmatrix}0&0\\1&0\end{pmatrix}$.

(2) Kac-Moody algebras

Idea is to take a Cartan matrix without the condition "positive definite" and to apply Serre construction.

So the ingredients:

1) Cartan matrix $A=(a_{ij})$, of rank l.
2) $\mathbb{C}$-vector space $\mathfrak{h}$ with $\dim(\mathfrak{h}) = 2n-l$.
3) n-linearly independent vectors $h_i \in \mathfrak{h}$, and n- . . . . . vector $\alpha_i \in \mathfrak{h}^*$, such that $\alpha_i(h_j) = a_{ji}$

<u>Receipt</u> The Kac-Moody algebra is given by generators e_i, f_i, $i=1,\dots n$, $h \in \mathfrak{h}$, satisfying.

$$(**)\left\{\begin{aligned}&[h,h']=0, \quad h,h' \in \mathfrak{h}\\&[h,e_i]=\alpha_i(h)e_i, \quad h\in\mathfrak{h}\\&[h,f_i]=-\alpha_i(h)f_i, \quad h\in\mathfrak{h} \qquad i,j\in 1\dots n\\&[e_i,f_j]=\delta_{ij}h_i\\&ad^{1-a_{ij}}(e_i)(e_j)=0\\&ad^{1-a_{ij}}(f_i)(f_j)=0\end{aligned}\right.$$

if A is:

a) positive-definied $\Longrightarrow$ simple Lie-algebras

b) positive-semi-defined $\longrightarrow$ affine Lie algebras

c) indefinite $\longrightarrow$ ~~inde~~ "zest" :).

③ Universal enveloping algebras

Ass - category of associative algebras

Lie - category of Lie algebras.

we have a functor $\mathcal{L}$: Ass $\longrightarrow$ Lie

$A \in$ Ass, it is Lie algebra with the bracket

$$[x,y] = xy - yx.$$

Let $\mathcal{U}$ be left adjoint functor to $\mathcal{L}$

$\mathcal{U}$: Lie $\longrightarrow$ Ass, and it has the following property for any

$$\begin{tikzcd} g \arrow[r, "\forall \varphi"] \arrow[d] & (A, [\cdot,\cdot]) \\ \mathcal{U}(g) \arrow[ur, dashed, "\exists ! \psi"'] & \end{tikzcd}$$

that is any representations of g extends to a representation $\mathcal{U}(g)$.

<u>Theorem</u> (PBW) Let g be a Lie algebra and let $x_1 \dots x_n \dots$ be a basis of g. Then the set

$$\{ x_{i_1} x_{i_2} \dots x_{i_r} \mid r \geq 0,\ i_1 \leq i_2 \leq \dots i_r \}$$ forms a basis of $\mathcal{U}(g)$

<u>Proposition</u> g Kac-Moody $\Longrightarrow \mathcal{U}(g)$ is isomorphic to an algebra generated by e_i, f_i, h_i subject to $(**)$ +

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \binom{1-a_{ij}}{l} e_i^l e_j e_i^{1-a_{ij}-l} = 0, \quad \sum_{l=0}^{1-a_{ij}} (-1)^l \binom{1-a_{ij}}{l} f_i^l f_j f_i^{1-a_{ij}-l}$$

(4) Quantum Kac-Moody algebras.

Let g be Kac-Moody algebra, then

$- \dots$ Deformation of $\mathcal{U}(g)$.

<u>Ingredients</u>

1) A - Generalized Cartan matrix
2) $(\mathfrak{h}, \mathfrak{h}^*, h_i, \alpha_i)$
3) v - formal variable, $\mathbb{C}(v)$ - field of rational functions

$$[n] := \frac{v^n - v^{-n}}{v - v^{-1}} = v^{-n} + v^{2-n} + \dots + v^{n-2} + v^n$$

$$[n]! := [n] \cdot [n-1] \dots [1].$$

$$\begin{bmatrix} t \\ r \end{bmatrix} := \frac{[t]!}{[r]!\,[t-r]!} = \frac{[t] \dots [t-r+1]}{[r] \dots [r-1][r]}.$$

$\mathcal{U}_v(g)$ defined as an algebra over $\mathbb{C}(v)$ generated by E_i, F_i, $i = 1 \dots n$, and v^h, for $h \in \mathfrak{h}$ with relations

$$\left.\begin{aligned} &v^h v^{h'} = v^{h+h'} \\ &v^h E_j v^{-h} = v^{\alpha_j(h)} E_j \\ &v^h F_j v^{-h} = v^{-\alpha_j(h)} F_j \end{aligned}\right\} \text{gen.}$$

$$[E_i, F_j] = \delta_{ij}\, [h_i].$$

$$\left.\begin{aligned} &\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix} E_i^l\, E_j\, E_i^{1-a_{ij}-l} = 0 \\ &\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix} F_i^l\, F_j\, F_i^{1-a_{ij}-l} = 0. \end{aligned}\right\} \text{q-Serre relations}$$

$i,j = 1..n$, $h \in \mathfrak{h}$.

By definition $U_q^+(\mathfrak{g})$ is a algebra generated by E_i

Ex. Simply-laced case $\Longleftrightarrow$ $A = (a_{ij})$ with $a_{ii} = 2$, $a_{ij} = 0, -1$ if $i \neq j$.

then out of q-Serre relations we have

$$\sum_{l=0}^{1} (-1)^l \begin{bmatrix} 1 \\ l \end{bmatrix} E_i^l E_j E_i^{1-l} = 0 \Rightarrow E_i E_j = E_j E_i \quad \text{if } a_{ij} = 0.$$

$$E_i^2 E_j - (q+q^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad \text{if } a_{ij} = -1.$$

Ex.