Hall algebras: Ringel theorem

Theorem (Ringel) there is an isomorphism

$$U_v^+(g_Q) \simeq H_{Rep_{\mathbb{F}_q} Q},$$

where Q is Dynkin quiver, q-prime, $v^2 = q$.

~~Recalling~~ Reminder

① Hall algebras $\mathbb{H}_{\mathcal{A}}$

$\mathcal{A}$ - is an abelian category, finitary.

$\mathcal{X} = Ob(\mathcal{A})/\sim$ - iso classes of objects in $\mathcal{A}$.

Hall algebra $\mathbb{H}_{\mathcal{A}}$ of $\mathcal{A}$ is as vector space spanned by elements of $\mathcal{X}$:

$$\mathbb{H}_{\mathcal{A}} = \bigoplus_{M \in \mathcal{X}} \mathbb{C}[M].$$

with multiplication

$$[M]\cdot[N] = \langle M,N\rangle_m \sum_{L\in\mathcal{X}} F_{M,N}^L [L],$$

where $\langle M,N\rangle_m = \left(\prod_{i=0}^{\infty} (\# Ext^i(M,N))^{(-1)^i}\right)^{1/2}$,

and $F_{M,N}^L$ is the number of elements in the set

$$F_{M,N}^L = \left|\{ R \subset L \mid R \simeq N, \cancel{} L/R \simeq M \}\right|$$

if A is k-linear then

$$\# \operatorname{Ext}^i(M,N) = q^{\dim \operatorname{Ext}^i(M,N)}, \quad \text{hence}$$

$$\langle M,N\rangle_m = \left(\frac{\# \operatorname{Hom}(M,N)}{\# \operatorname{Ext}(M,N)} \cdot \# \operatorname{Ext}^2(M,N) \cdots \right)^{-1/2} =$$

$$= q^{\frac{1}{2}\sum_{i=1}^{\infty}(-1)^i \dim \operatorname{Ext}^i(M,N)}.$$

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② Let Q be a quiver, i.e. $Q = (Q_0, Q_1)$
vertices, arrows.

The category $\mathrm{Rep}_k Q$ has objects

$$X = \Big((X_i)_{i \in Q_0}, (X_\beta : X_i \to X_j)_{\beta \in Q_1} \Big)$$

k-vector spaces, k-linear maps

$\operatorname{Hom}(X,Y)$ - is a collection of k-linear maps
$(f_i : X_i \to Y_i)_{i \in Q_0}$, such that

$$\begin{tikzcd} X_i \arrow[r, "X_\beta"] \arrow[d, "f_i"'] & X_j \arrow[d, "f_j"] \\ Y_i \arrow[r, "Y_\beta"'] & Y_j \end{tikzcd}$$

commute for all $\beta \in Q_1$,
$Y_\beta \circ f_i = f_j \circ X_\beta$, $\beta : i \to j$.

$\mathrm{Rep}_k Q$ - is hereditary, that is $\operatorname{Ext}^i(X,Y) = 0$, if $i > 1$.

Hence $\langle X,Y\rangle = q^{\frac{1}{2}(\dim \operatorname{Hom}(X,Y) - \dim \operatorname{Ext}(X,Y))}$, if $k = \mathbb{F}_q$

the simple objects are given by

$$(S_i)_j = \begin{cases} k, & i=j \\ 0 & i\neq j. \end{cases} \quad S_\beta = 0$$

So simples has the form

$$\begin{tikzcd}
& & & & 0 \arrow[dl] & \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & k & 0 \arrow[l] & 0 \arrow[l] \\
& & 0 \arrow[ur] & & 0 \arrow[ul] &
\end{tikzcd}$$

Also we have:

$$\mathrm{Hom}(S_i, S_j) = \begin{cases} k, & \text{if } i=j \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathrm{Ext}(S_i, S_j) = k^{\#\{\beta : i \to j\}}$$

Theorem (Gabriel) Q has finite type $\Leftrightarrow$ Q is Dynkin, def. $\Leftrightarrow$ A_Q is positive def.

The map $X \mapsto (\dim X_i)_{i\in Q_0}$ is a bijection between indecomposables and roots

in particular we have

$$\langle S_i, S_j \rangle_m = \begin{cases} q^{\frac{1}{2}} & \text{if } i=j \\ q^{-\frac{1}{2}\#\text{arrows}} & i\neq j. \end{cases}$$

③ $U_{q,v}(g_Q)$.

We build the Cartan matrix out of Q.

$$A_Q = (a_{ij})_{i,j=1}^{|Q_0|}, \quad a_{ij} = \begin{cases} 2 & i=j \\ 0 & \text{if there exist no arrows } i\to j, \text{ or } j \to i \\ -1 & \text{otherwise.} \end{cases}$$

v - be a formal variable, and let $\mathbb{C}(v)$ be a field of rational functions on v

$U_v^+(g_Q)$ is a $\mathbb{C}(v)$-algebra generated by $E_1 \dots E_n$, $n = |Q_0|$ with relations:

$$E_iE_j - E_jE_i = 0 \quad \text{if } a_{ij}=0$$

$$E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0, \quad \text{if } a_{ij}=-1.$$

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(4) "Proof" (construction).

the correspondence $E_i \to [S_i]$

generates an isomorphism: ψ

1) ψ is a morphism of algebras

$$[S_i][S_j] = [S_j][S_i], \quad \text{if } a_{ij}=0$$

$$[S_i]^2[S_j] - (v+v^{-1})[S_i][S_j][S_i] + [S_j][S_i]^2 = 0.$$

2) ψ is injective.

3) ψ is surjective, that is $[S_i]$ generates $\mathbb{H}_{\mathrm{Rep}_{\mathbb{F}_q}Q}$

1)–2) hold in general, 3) fails in affine case.

1). assume that $a_{ij}=0$, then there is no arrows between i and j, and no nontrivial extensions between S_i and S_j $\Rightarrow$ $\langle S_i, S_j\rangle_m = \langle S_j, S_i\rangle_m =$

$$= q^{\frac{1}{2}\dim \mathrm{Hom} - \dim \mathrm{Ext}} = q^0 = 1.$$

$$[S_i][S_j] = [S_i \oplus S_j] = [S_j][S_i].$$

assume that $a_{ij}=-1$, say there is an arrow $i \to j$, then $\mathrm{Ext}^1(S_i, S_j) = \mathbb{F}_q$

$$\mathrm{Ext}^1(S_j, S_i) = 0$$

We have one non-trivial extension

$$0 \to S_j \to I_{ij} \to S_j \to 0$$

$$S_i \quad \overset{k}{\bullet} \longrightarrow 0 \qquad I_{ij}: k \xrightarrow{1} k$$

$$S_j \quad 0 \longrightarrow k,$$

Hence we have $\langle S_1, S_2 \rangle_m = q^{-\frac{1}{2}} = v^{-1}$

$$[S_i]\cdot[S_j] = v^{-1}\left([S_i \oplus S_j] + [I_{ij}]\right).$$

On other hand we have that $\langle S_j, S_i \rangle_m = 1$.

$$[S_j]\cdot[S_i] = [S_j \oplus S_i].$$

$$\boxed{[S_j]\cdot[S_i]^2 = v(v^2+1)[S_j^{\oplus}][S_i^{\oplus 2}] = [S_j^{\oplus 2} \oplus S_i].}$$

$$[S_i]^2 = v(v^2+1)\,[S_i \oplus S_i]$$

$$\boxed{\begin{aligned}[S_i]^2\cdot[S_j] &= v(v^2+1)[S_i^{\oplus 2}][S_j] = \\ &= v^{-1}(v^2+1)\left([S_i^{\oplus 2} \oplus S_j] + [S_i \oplus I_{ij}]\right)\end{aligned}}$$

$$\mathrm{Hom}(S_j, S_i^{\oplus 2} \oplus S_{jj}) = \mathrm{Hom}(S_j, S_i \oplus I_{ij}) = k.$$

$$[S_i]\cdot[S_j]\cdot[S_i] = (v^2+1)[S_i^{\oplus 2} \oplus S_j] + [S_i \oplus I_{ij}]$$

$$[S_i]^2\cdot[S_j] - (v+v^{-1})[S_i][S_j][S_i] + [S_j][S_i]^2 =$$

$$= \underline{v^{-1}(v^2+1)\cdot[S_i^{\oplus 2} \oplus S_j]} + \boxed{v^{-1}(v^2+1)[S_i \oplus I_{ij}]} - \underline{(v+v^{-1})(v^2+1)[S_i^{\oplus 2} \oplus S_i]} -$$

$$\boxed{-(v+v^{-1})[S_i \oplus I_{ij}]} + \underline{v(v^2+1)[S_j^{\oplus 2} + S_i]} = 0$$