NOTES FOR "QUANTUM GROUPS, CRYSTAL BASES AND REALIZATION OF $\widehat{\mathfrak{sl}}(n)$-MODULES"

EVAN WILSON

1. Quantum groups

Consider the Lie algebra $\mathfrak{sl}(n)$, which is the Lie algebra over $\mathbb{C}$ of $n \times n$ trace 0 matrices together with the commutator bracket: $[A, B] = AB - BA$. It is easy to verify that this is generated by the elements $e_i := E_{i,i+1}, f_i := E_{i+1,i}, h_i := E_{i,i} - E_{i+1,i+1}, i = 1, 2, \ldots n-1$ where $E_{i,j}$ is the matrix with 1 in the ith row and jth column and 0 everywhere else and satisfies the following relations:

(1) $[h_i, h_j] = 0, i, j = 1, 2, \ldots, n-1$,
(2) $[e_i, f_j] = \delta_{ij} h_i, i, j = 1, 2, \ldots, n-1$,
(3) $[h_i, e_j] = a_{ij} e_j, i, j = 1, 2, \ldots, n-1$
(4) $[h_i, f_j] = -a_{ij} f_j, i, j = 1, 2, \ldots, n-1$
(5) $(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) = 0, (\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j) = 0$, for $i \neq j$,

where $A = (a_{ij})_{i,j=1}^{n-1}$ is the matrix:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}$$

and $\mathrm{ad}_x(y) := [x, y]$ is the adjoint operator.

In fact, $\mathfrak{sl}(n)$ is isomorphic to the Lie algebra generated by $e_i, f_i, h_i, i = 1, 2, \ldots, n$ subject to relations (1)-(5) (see for example [H]). This is our motivating example for the notion of Kac-Moody algebras, which we define next. A standard reference is [K].

Every Kac-Moody algebra is determined by a *generalized Cartan matrix* (GCM), which is a matrix $A = (a_{ij})_{i,j \in I}, a_{ij} \in \mathbb{Z}$, where I is a finite index set, satisfying the following conditions:

(1) $a_{ii} = 2$,
(2) $a_{ij} \leq 0$ if $i \neq j$,
(3) $a_{ij} < 0$ if and only if $a_{ji} < 0$.

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A GCM is called *symmetrizable* if there exists a diagonal matrix $D = \mathrm{diag}(s_i)_{i\in I}$ such that $s_i \in \mathbb{Z}_{>0}, i \in I$ and DA is a symmetric matrix. We will consider only symmetrizable GCMs. It may happen that A is singular. Fix a subset S of I of cardinality $|I| - \mathrm{rank}(A)$.

Definition 1. *Let $A = (a_{ij})_{i,j\in I}$ be a (symmetrizable) GCM. The* Kac-Moody algebra $\mathfrak{g}(A)$ *is the Lie algebra over $\mathbb{C}$ generated by the elements $e_i, f_i, i \in I$, and $d_j, j \in S$ satisfying the following relations:*

(1) $[h_i, h_j] = 0, i, j \in I, [h_i, d_j] = 0, i \in I, j \in S, [d_i, d_j] = 0, i, j \in S$
(2) $[e_i, f_j] = \delta_{ij} h_i, i, j \in I$
(3) $[h_i, e_j] = a_{ij} e_j, i, j \in I, [d_i, e_j] = \delta_{ij} e_j, i \in I, j \in S$
(4) $[h_i, f_j] = -a_{ij} f_j, i, j \in I, [d_i, f_j] = -\delta_{ij} f_j, i \in I, j \in S$
(5) $(ad_{e_i})^{1-a_{ij}}(e_j) = 0, (ad_{f_i})^{1-a_{ij}}(f_j) = 0,$ *for $i \neq j \in I$,*

Where there is no confusion about A, we write $\mathfrak{g}$ for $\mathfrak{g}(A)$.

We remark that the Kac-Moody algebra $\mathfrak{g}(A)$ is finite dimensional if and only if A is a positive definite matrix. In this case, it is a finite dimensional semisimple Lie algebra (see [H]).

We recall also the notion of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ which is the unique associative algebra $U(\mathfrak{g})$ equipped with a map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ such that $\iota([x, y]) = \iota(x)\iota(y) - \iota(y)\iota(x)$ which satisfies the following universal property: if $\mathcal{A}$ is any associative algebra and $\kappa : \mathfrak{g} \to \mathcal{A}$ is a map satisfying $\kappa([x, y]) = \kappa(x)\kappa(y) - \kappa(y)\kappa(x)$ then there exists a unique homomorphism of algebras $\phi : U(\mathfrak{g}) \to \mathcal{A}$ such that $\phi \circ \iota = \kappa$, alternately, such that the following diagram commutes:

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[d, dashed, "!\phi"] \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[dr, "\kappa"'] & \\ & \mathcal{A} \end{tikzcd}$$

Recall also (by the Poincaré-Birkhoff-Witt theorem) that $\mathfrak{g}$ embeds via ι isomorphically in $U(\mathfrak{g})$, which can be realized as a suitable quotient of the tensor algebra of $\mathfrak{g}$. In order to quantize the universal enveloping algebra of a Kac-Moody algebra, it is helpful to give an explicit realization in terms of generators and relations:

Proposition 1 (see [HK])**.** *Let $\mathfrak{g}$ be a Kac-Moody algebra with GCM A. Then $U(\mathfrak{g})$ is isomorphic to the associative algebra over $\mathbb{C}$ with 1 generated by $e_i, f_i, h_i, i \in I, d_j, j \in S$ satisfying relations (1)-(4) of definition 1 (with Lie bracket replaced by commutator bracket) and the following:*

(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for $i \neq j$,*

(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for $i \neq j$.*

Now we are ready to give the definition of a quantum group. Let $\mathbb{C}(q)$ be the field of rational functions in the indeterminate q and define $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} \in \mathbb{C}(q)$ to be the q integer m. Define the q-factorial inductively by $[0]_q! = 1, [m]_q! = [m]_q[m]_q!$. Finally, define $\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$ to be the q binomial coefficient. Notice that the limit as $q \to 1$ of each of these functions is $m, m!$ and $\binom{m}{n}$ respectively. Then we make the following definition:

Definition 2. *Let $\mathfrak{g}$ be a Kac-Moody algebra with GCM A. The* quantum group *or* quantized universal enveloping algebra $U_q(\mathfrak{g})$ *is the associative algebra over $\mathbb{C}(q)$ with 1 generated by the elements e_i, f_i, K_i, K_i^{-1} $i \in I$ and $D_j, D_j^{-1}, j \in S$ with the following defining relations:*

(1) $K_iK_i^{-1} = K_i^{-1}K_i = D_jD_j^{-1} = D_j^{-1}D_j = 1, i \in I, j \in S$
(2) $[K_i, K_j] = 0, i, j \in I, [K_i, D_j] = 0, i \in I, j \in S, [D_i, D_j] = 0, i, j \in S,$
(3) $K_ie_jK_i^{-1} = q^{a_{ij}}e_j, i, j \in I, D_ie_jD_i^{-1} = q^{\delta_{ij}}e_j, i \in I, j \in S$
(4) $K_if_jK_i^{-1} = q^{-a_{ij}}f_j, i, j \in I, D_if_jD_i^{-1} = q^{-\delta_{ij}}e_j, i \in I, j \in S$
(5) $[e_i, f_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ *for* $i, j \in I,$
(6) $\sum_{k=0}^{1-a_{ij}}(-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} e_i^{1-a_{ij}-k}e_je_i^k = 0$ *for* $i \neq j,$
(7) $\sum_{k=0}^{1-a_{ij}}(-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} f_i^{1-a_{ij}-k}f_jf_i^k = 0$ *for* $i \neq j.$

Where $q_i = q^{s_i}$.

It is possible to view $U(\mathfrak{g})$ as the formal limit as $q \to 1$ of $U_q(\mathfrak{g})$, hence the terminology (see [HK]). There is also an analogous theory of $U_q(\mathfrak{g})$-modules with parallels to the theory of $U(\mathfrak{g})$-modules (equivalently representations of $\mathfrak{g}$). We outline the main points next.

Let $\lambda \in P := \{\lambda \in \mathfrak{h}^* | \lambda(h_i), \lambda(d_j) \in \mathbb{Z}, i \in I, j \in S\}$, where $\mathfrak{h} = \text{span}_{\mathbb{C}}\{h_i, i \in I, d_j, j \in S\}$ is the Cartan subalgebra of $\mathfrak{g}$. Then there exists a $U_q(\mathfrak{g})$-module $V^q(\lambda)$ which is an irreducible highest weight module with highest weight λ i.e., there exists $v_\lambda \in V^q(\lambda)$ such that $V^q(\lambda) = U_q(\mathfrak{g})v_\lambda$ and

(1) $e_iv_\lambda = \{0\}, i \in I$
(2) $K_i \cdot v_\lambda = q^{\lambda(h_i)}v_\lambda, i \in I, D_i \cdot v_\lambda = q^{\lambda(d_j)}v_\lambda, j \in S.$

For $\mu \in P$ we define the set $V^q(\lambda)_\mu := \{v \in V^q | K_iv = q^{\mu(h_i)}v, i \in I, D_jv = q^{\mu(d_j)}, j \in S\}$ to be the *weight space* of $V^q(\lambda)$ of weight μ. One of the main results is the following, due to Lusztig:

Theorem 1. *Let $\lambda \in P^+ := \{\lambda \in P | \lambda(h_i), \lambda(d_i) \geq 0, i \in I, j \in S\}$. Then*

$$dim_{\mathbb{C}(q)}(V^q(\Lambda)_\mu) = dim_{\mathbb{C}}(V(\lambda)_\mu), \mu \in P.$$

where $V(\lambda)$ is the irreducible highest-weight $U(\mathfrak{g})$-module with highest weight λ.

Therefore, the characters of integrable $U_q(\mathfrak{g})$-modules are the same as the corresponding $U(\mathfrak{g})$-modules.

2. Crystal Bases

Crystal bases ([Ka],[Lu]) are essentially bases of $U_q(\mathfrak{g})$ modules in the formal limit as $q \to 0$. Before we define crystal bases, we need the notion of the *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i, i \in I$. These are certain modified root vectors for the quantum group $U_q(\mathfrak{g})$. But first, we need a preliminary result:

Lemma 1 ([Ka])**.** *Let $\lambda \in P^+$ and $V^q(\lambda)$ be the highest weight $U_q(\mathfrak{g})$-module of highest weight λ. For each $i \in I$, every weight vector $u \in V^q(\lambda)_\mu (\mu \in P)$ may be written in the form*

$$u = u_0 + f_i u_1 + \cdots + f_i^{(N)} u_N,$$

where $N \in \mathbb{Z}_{\geq 0}$ and $u_k \in V^q(\lambda)_{\mu + k\alpha_i} \cap \ker e_i$ for all $k = 0, 1, \ldots, N, \alpha_i \in P$ is defined by $\alpha_i(h_j) = a_{ji}, i, j \in I, \alpha_i(d_j) = \delta_{i,j}, i \in I, j \in S$, and $f_i^{(n)} := \frac{f_i^n}{[n]_{q_i}!}$. Here, each u_k in the expression is uniquely determined by u and $u_k \neq 0$ only if $\mu(h_i) + k \geq 0$.

We now have the following:

Definition 3. *Let $\lambda \in P^+$. The* Kashiwara operators *$\tilde{e}_i$ and $\tilde{f}_i (i \in I)$ on $V^q(\lambda)$ are defined by*

$$\tilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k.$$

We also need an auxiliary definition of a *crystal lattice*, which will make it possible to formally take the limit as $q \to 0$. Let $\mathbb{A}_0 := \{f \in \mathbb{C}(q) | f \text{ is regular at } 0\}$.

Definition 4. *Let $\lambda \in P^+$ and $V^q(\lambda)$ be the highest weight $U_q(\mathfrak{g})$-module of highest weight λ. A free $\mathbb{A}_0$-submodule $\mathcal{L}$ of $V^q(\lambda)$ is called a* crystal lattice *if*

1. *$\mathcal{L}$ generates $V^q(\lambda)$ as a vector space over $\mathbb{C}(q)$,*
2. *$\mathcal{L} = \bigoplus_{\mu \in P} \mathcal{L}_\mu$, where $\mathcal{L}_\mu = \mathcal{L} \cap V^q(\lambda)_\mu$,*
3. *$\tilde{e}_i \mathcal{L} \subset \mathcal{L}, \tilde{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$.*

Finally, we have the following:

Definition 5. *A* crystal base *of the irreducible highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda), \lambda \in P^+$ is a pair $(\mathcal{L}, \mathcal{B})$ such that*

1. *$\mathcal{L}$ is a crystal lattice of $V^q(\lambda)$,*
2. *$\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L}$,*
3. *$\mathcal{B} = \bigsqcup_{\mu \in P} \mathcal{B}_\mu$, where $\mathcal{B}_\mu = \mathcal{B} \cap (\mathcal{L}_\mu / q\mathcal{L}_\mu)$,*
4. *$\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$ for all $i \in I$,*
5. *for any $b, b' \in \mathcal{B}$ and $i \in I$, we have $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.*

The set $\mathcal{B}$ is called the *crystal* or *crystal graph* of $(\mathcal{L}, \mathcal{B})$. This is because $\mathcal{B}$ can be regarded as a colored, oriented graph by defining

$$b \xrightarrow{i} b' \iff \tilde{f}_i b = b'.$$

Example: Let $V^q(m), m \geq 0$ be the highest weight $U_q(\mathfrak{sl}(2))$-module $\mathrm{span}_{\mathbb{C}(q)}\{v_0, v_1, \ldots, v_m\}$ with the following actions of the generators:

$$\begin{aligned} K \cdot v_j &= q^{m-2j} v_j \\ f \cdot v_j &= [j+1]_q v_{j+1} \\ e \cdot v_j &= [m-j+1]_q v_{j-1} \end{aligned}$$

where v_j is understood to be 0 if $j < 0$ or $j > m$. Then $\mathcal{L} = \mathrm{span}_{\mathbb{A}_0}\{v_j | j = 0, 1, \ldots, m\}$ is a crystal lattice of $V^q(m)$ and $(\mathcal{L}, \mathcal{B})$ is a crystal base, where $\mathcal{B} = \{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_m\}$ (we write $\bar{v}_i$ for $v_i + q\mathcal{L}$). The action of the Kashiwara operators on $\mathcal{B}$ is

$$\begin{aligned} \tilde{e}(\bar{v}_j) &= \bar{v}_{j-1}, \\ \tilde{f}(\bar{v}_j) &= \bar{v}_{j+1}, \end{aligned}$$

again $\bar{v}_j$ is understood to be 0 if $j < 0$ or $j > m$. Therefore the crystal graph is:

$$\bar{v}_0 \to \bar{v}_1 \to \bar{v}_2 \to \cdots \to \bar{v}_m$$

In this way, theory of crystal bases enables us to study "combinatorial skeleton" of $U_q(\mathfrak{g})$-modules. In particular, the crystal base of a $U_q(\mathfrak{g})$-module satisfies the conditions for an *(abstract) crystal.*

Definition 6. *A* crystal *associated with $U_q(\mathfrak{g})$ is a set $\mathcal{B}$ together with maps $wt : \mathcal{B} \to P, \tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$, and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$, for $i \in I$ satisfying the following properties:*

(1) *$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle$ for all $i \in I$,*
(2) *$wt(\tilde{e}_i b) = wt(b) + \alpha_i$ if $\tilde{e}_i b \in \mathcal{B}$,*
(3) *$wt(\tilde{f}_i b) = wt(b) - \alpha_i$ if $\tilde{f}_i b \in \mathcal{B}$,*
(4) *$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in \mathcal{B}$,*
(5) *$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in \mathcal{B}$,*
(6) *$\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in \mathcal{B}, i \in I$,*
(7) *if $\varphi_i(b) = -\infty$ for $b \in \mathcal{B}$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.*

Then one may easily prove the following:

Proposition 2 ([Ka])**.** *Let $(\mathcal{L}, \mathcal{B})$ be the crystal basis of a $U_q(\mathfrak{g})$-module $V^q(\lambda)$. Then $\mathcal{B}$ is a crystal if we define in addition to $\tilde{e}_i, \tilde{f}_i$:*

- *$wt(b) = \mu$ if $b \in \mathcal{B}_\mu$,*
- *$\varepsilon_i(b) = \max\{k | \tilde{e}_i^k(b) \neq 0\}$,*
- *$\varphi_i(b) = \max\{k | \tilde{f}_i^k(b) \neq 0\}$.*

i	$i+1$	$i+2$	$\cdots$
$i-1$	i	$i+1$	$\cdots$
$i-2$	$i-1$	i	$\cdots$
$\vdots$	$\vdots$	$\vdots$	

FIGURE 1. Color pattern for an extended Young diagram of charge i. All labels are reduced modulo n.

3. REALIZATION OF $\widehat{\mathfrak{sl}}(n)$-MODULES

We probably should say "combinatorial realization of $U_q(\widehat{\mathfrak{sl}}(n))$-modules", but in light of Theorem 1 the terminology is justified. The Kac-Moody algebra $\widehat{\mathfrak{sl}}(n)$ has GCM $A = (a_{i,j})_{i,j\in I}$ where $I = \{0, 1, \dots, n\}$ and

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

The Kac-Moody algebra $\widehat{\mathfrak{sl}}(n)$ is the infinite-dimensional analogue of $\mathfrak{sl}(n)$, hence the terminology.

Let $I := \{0, 1, \dots n-1\}$ denote the index set for $\widehat{\mathfrak{sl}}(n)$. An *extended Young diagram* is a collection of I-colored boxes arranged in rows and columns, such that the number of boxes in each row is greater than or equal to the number of boxes in the row below. To every extended Young diagram we associate a *charge*, $i \in I$. In each box, we put a *color* $j \in I$ given by $j \equiv a - b + i \pmod{n}$ where a is the number of columns from the right and b is the number of rows from the top (see figure 1).

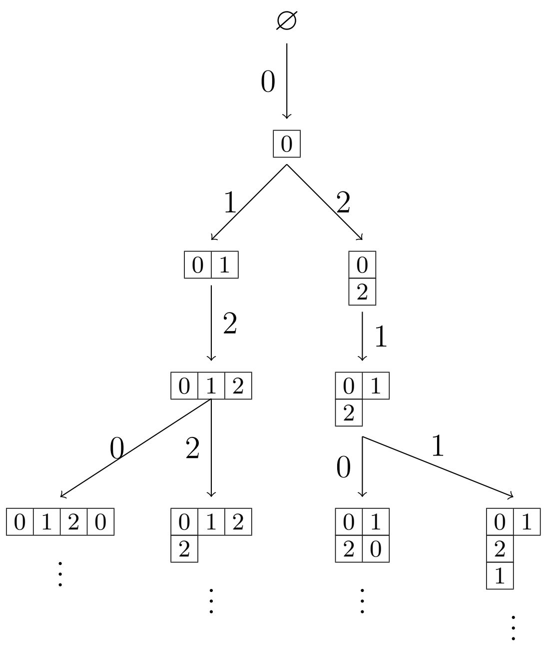
For example, $\begin{array}{|c|c|c|}\hline 1&2&0\\ \hline 0&1\\ \cline{1-2}\end{array}$ is an extended Young diagram of charge 1 for $n = 3$. The *null diagram* with no boxes—denoted by $\varnothing$—is also considered as an extended Young diagram.

A column in an extended Young diagram is *i-removable* if the bottom box contains i and can be removed leaving another extended Young diagram. A column is *i-admissible* if a box containing i could be added to give another extended Young diagram.

An extended Young diagram is called *n-regular* if there are at most $(n-1)$ rows with the same number of boxes. Let $\mathcal{Y}(i), i \in I$ denote the collection of all n-regular extended Young diagrams of charge j. Then $\mathcal{Y}(i)$ can be given the structure of a crystal with the following actions of $\tilde{e}_j$, $\tilde{f}_j$, ε_j, φ_j, and $\mathrm{wt}(\cdot)$. For each $j \in I$ and $b \in \mathcal{Y}(i)$ we define the *j-signature* of b to be the string of $+$'s, and $-$'s in which each j-admissible column receives a $+$ and each j-removable column receives a $-$ reading from right to left. The *reduced j-signature* is the result of recursively canceling all '$+-$' pairs in the i-signature leaving a string of the form $(-,\dots,-,+\dots,+)$. The Kashiwara operator $\tilde{e}_j$ acts on b by removing the box corresponding to the rightmost $-$, or maps b to 0 if there are no minus signs. Similarly, $\tilde{f}_j$ adds a box to the bottom of the column corresponding to the leftmost $+$, or maps b to 0 if there are no plus signs. The function $\varphi_j(b)$ is the number of $+$ signs in the reduced j-signature of b and $\varepsilon_j(b)$ is the number of $-$ signs. We define wt: $\mathcal{Y}(i) \to P$ by $b \mapsto \Lambda_i - \sum_{j=0}^{n-1} \#\,\{j\text{-colored boxes in } b\}\alpha_j$ where $\Lambda_i \in P$ is the ith fundamental weight given by $\Lambda_i(h_j) = \delta_{ij}, \Lambda_i(d) = 0$. Then we have the following:

Theorem 2 (MM)**.** *Let $\mathcal{B}(\Lambda_i)$ be the crystal graph of the $U_q(\widehat{\mathfrak{sl}}(n))$-module $V^q(\Lambda_i)$. Then $\mathcal{Y}(i) \cong \mathcal{B}(\Lambda_i)$ as crystals.*

We give the top part of $\mathcal{Y}(0)$ for $\widehat{\mathfrak{sl}}(n)$ below:



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