Nakajima's quiver varieties

Categorical quocients

⓪ Why?

a) Our motivation: the construction of Nakajima's quiver variety involves this notion. More presicely the notion of moduli space for quivers.

b) General motivation: Given variety X (affine or projective) and algebraic group action $G \circlearrowright X$. To form a quocient X/G in such a way that resulting variety will lie in the same category (i.e. X/G is affine or projective). Usual orbit-space does not work here as the following example shows.

Example 0. $X = \mathbb{C}^n$, $G = \mathbb{C}^*$. Then X/G is not Hausdorf (!).

The aim of this lecture is to give brief introduction to GIT which allows to avoid such situation.

① Affine case

a) Affine varieties. We fix one and forever field $\mathbb{C}$.

Def. An affine variety X in $\mathbb{C}^n$ is a common zero-locos of some collection of polinomials $f_i \in \mathbb{C}[x_1, \ldots x_n]$, i.e

$$X = \{(x_1, \ldots x_n) \in \mathbb{C}^n \mid f_i(x_1, \ldots x_n) = 0,\ i \in I\}.$$

It is not hard to see that there exist 1-1 corresp. between:

$$\text{ideal } I \subset \mathbb{C}[x_1 \dots x_n] \longleftrightarrow \begin{array}{c}\text{affine varieties}\\ X \subset \mathbb{C}^n.\end{array}$$

Under this correspondence

$$\begin{array}{c}\text{maximal ideals}\\ \text{in } \mathbb{C}[x_1 \dots x_n]\end{array} \longleftrightarrow \text{points in } \mathbb{C}^n.$$

Which are the consequenses of Hilbert theorems:

1) every ideal in $\mathbb{C}[x_1 \dots x_n]$ is finitely generated.

2) every maximal ideal has a form $(x_1 - a_1, \dots, x_n - a_n)$ for some $(a_1 \dots a_n) \in \mathbb{C}^n$.

6) <u>Spectrum of the ring</u>

<u>Def.</u> Let R be a ring. <u>Spectrum</u> of R, denoted by $\operatorname{Spec} R$ is set of all maximal ideals in R.

<u>Example 1</u> As we saw $\operatorname{Spec} \mathbb{C}[x_1 \dots x_n] \simeq \mathbb{C}^n$.

<u>Example 2</u> Let I be an ideal in $\mathbb{C}[x_1 \dots x_n]$, then $\operatorname{Spec} \mathbb{C}[x_1 \dots x_n]/I \simeq X_I$ - an affine variety determined by I.

Given any ring R, $\operatorname{Spec} R$ - affine variety, and the other way around Given an affine variety $X \subset \mathbb{C}^n$, there exist such R that $\operatorname{Spec} R$.

Namely R - is a coordinate ring $\mathbb{C}[x_1 \dots x_n]/I$.

c) Group action

Suppose that complex Lie group G acts on R by ring automorfism

Example $R=\mathbb{C}[x\ldots x_n]$, $G=GL_n(\mathbb{C})$ acts by linear changes of x_i's

G takes maximal ideal to maximal, therefore G acts on Spec R. We are interesting in forming quocient $(\mathrm{Spec}\,R)/G$. As we are in category of affine varieties we are loking for some ring S, such that $\mathrm{Spec}\,S = \mathrm{Spec}\,R/G$, or equivalently S is the ring of function on $\mathrm{Spec}\,R/G$. But then each element of S pulls back to G-invariant function on Spec R. Retone by R^G – the ring of G-invariant functions on Spec R.

Def. We call $\mathrm{Spec}\,(R^G)$ – the geometric invariant theory quocient (or GIT) of Spec R by G. And denote it by $(\mathrm{Spec}\,R)/\!/G$.

Obs 1) It not quite true that $(\mathrm{Spec}\,R)/G = (\mathrm{Spec}\,R)/\!/G$. but there is a map $(\mathrm{Spec}\,R)/G \to (\mathrm{Spec}\,R)/\!/G$. Suposse that G is reductive then

2) R^G is finitely generated (By Hilbert's theorem).

3) $(\operatorname{Spec} R)/G \to (\operatorname{Spec} R)//G$ is onto.

<u>Example 3</u> Let $G = \mathbb{C}^*$ acts on $R = \mathbb{C}[x_1, \dots x_n]$ by rescaling each coordinate. Then G acts on $\operatorname{Spec} R = \mathbb{C}^n$ by rescaling as well, the only invariant polinomials are the constant $\Rightarrow R^G = \mathbb{C}$. But then $\operatorname{Spec} \mathbb{C} = pt$.

② <u>Projective case</u>

a) <u>Projective spectrum</u>

<u>Def.</u> A <u>projective variety</u> X in $\mathbb{CP}^n$ is a common-zero locus of some collection of homogeneous polinomials $f_i \in \mathbb{C}[x_0, \dots x_n]$, i.e.

$$X = \{ (x_0, \dots, x_n) \mid f_i(x_0, \dots, x_n) = 0 \}$$

Let R be <u>graded</u> ring. $R = \bigoplus_{k \in \mathbb{N}} R_k$, i.e. $R_k \cdot R_n \subseteq R_{k+n}$.

For example $\mathbb{C}[x_1 \dots x_n]$ is graded, with R_k - hom. polynomials of degree k.

Several equivalent definitios of <u>projective spectrum</u> of R, $\operatorname{Proj} R$:

1. $\operatorname{Proj} R$ – the set of maximal graded ideals of R, i.e. when $I = \bigoplus_{n\in\mathbb{N}} I_n$, and $I_n = I \cap R_n$.

2. $\operatorname{Proj} R = (\operatorname{Spec} R \setminus \operatorname{Spec} R_0)/\mathbb{C}^*$

<u>Example 4</u> If $R = \mathbb{C}[x_1 \dots x_n]$. then $R_0 = \mathbb{C}$. and

$$\operatorname{Proj} R = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^* \simeq \mathbb{CP}^{n-1}$$

3. If R is presented as $\mathbb{C}[x_1 \dots x_n]/I$. then $\mathbb{C}[x_1 \dots x_n] \twoheadrightarrow R$ or dualy $\operatorname{Spec} R \hookrightarrow \mathbb{C}^n$ usualy gives $\operatorname{Proj} R \hookrightarrow \mathbb{CP}^{n-1}$ in this case $\operatorname{Proj} R$ is a projetive variety.

Another <u>relation</u> between Spec and Proj

<u>Relation</u> Given ungraded ring R_0 we define $R := R_0[e]$ with deg $e = 1$. Then $\operatorname{Spec} R = \operatorname{Spec} R_0 \times \mathbb{C}$. Hence $(\operatorname{Spec} R \setminus \operatorname{Spec} R_0) = \operatorname{Spec} R_0 \times \mathbb{C}^*$ and so

$$\operatorname{Proj} R = (\operatorname{Spec} R \setminus \operatorname{Spec} R_0)/\mathbb{C}^* = \operatorname{Spec} R_0 .$$

<u>Obs</u>. Given a map $R \to S$ we have a map

$$\operatorname{Spec} S \to \operatorname{Spec} R$$

<u>Expectation</u>: if $f: R \to S$ is hom. of graded rings then $f: \operatorname{Proj} S \to \operatorname{Proj} R$. But(!).

<u>Proposition</u> Let $f: R \to S$ be hom of graded rings. Let X be the points of $\operatorname{Spec} S \setminus \operatorname{Spec} S_0$ such that every function in $f(R)$ vanish on X. Let $\operatorname{Proj}^{us}_f = X/\mathbb{C}^*$. Then f induces a map $\operatorname{Proj} S \setminus (\operatorname{Proj} S)^{us}_f \to \operatorname{Proj} R$

We will call the set $(\mathrm{Proj}\, S)^{us}_f$ unstable.

b) GIT.

Given a graded ring R with an action of G we define GIT quotient as

$$(\mathrm{Proj}\, R)//G := \mathrm{Proj}\,(R^G) \text{ "}=\text{" } (\mathrm{Proj}\, R)^S / G$$

this notation $\mathrm{Proj}\, R // G$ is misleading (!) due to:

1). ~~Bo~~ Usualy one starts with the variety $\mathrm{Proj}\, R = X$ and G-action there. And there is a freedom in chosing ring R such that $\mathrm{Proj}\, R = X$.

2) Chosing the ring, there is a freedom in chosing group action on it such that leads to the same group action on $\mathrm{Proj}\, R$.

1)-2) - Leads to diferent stability condition of the group action on variety.
Usualy one consider the line bundle $X \times \mathbb{C}$, equiping it with the action of G. Depending on character $\chi : G \to \mathbb{C}^*$, there is another freedom here.

Example 4. Let $\mathbb{C}^n = X$, $\mathbb{C}^* = G$. Then $X = \mathrm{Spec}\,\mathbb{C}[x_1 \dots x_n] =$
$\mathrm{Proj}\,\mathbb{C}[x_1 \dots x_n, e]$

The action of G on $\mathbb{C}[x_1 \dots x_n, e]$ is not quite determine
if $z \in \mathbb{C}^*$, then $z * x_i := z x_i$. But
$$z * e = z^k \cdot e.$$

Therefore we have the following:

1) if $k>0$, then there are no invariant polynomials other than constant: $R^G = \mathbb{C}$, and $\operatorname{Proj} R^G$ – is empty.

2) if $k=0$, then $R^G = \mathbb{C}[\ell]$, and $\operatorname{Proj} R^G = pt$.

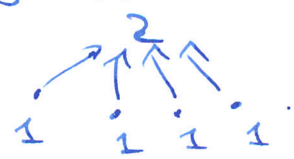
3) if $k \leq -1$ the invariant ring $R^G = \mathbb{C}[x_1\ell, x_2\ell, \dots, x_n\ell]$. Hence $\operatorname{Proj} R^G = \mathbb{CP}^{n-1}$. this hold's for any $k<0$.

<u>Example</u> 5 Let $X = \mathbb{C}^{m\times n}$, the space of $m\times n$ matrices with $m\leq n$. $G = GL_m(\mathbb{C})$ act on X by left multiplication.

We regard X as the space of m vectors in $\mathbb{C}^n$. If GIT $X/\!/G$ is non empty then $X/\!/G$ is just $Gr_m(\mathbb{C}^n) = \{\text{subspaces of dim } m \text{ in } \mathbb{C}^n\}$.

<u>Example</u> 6. Let Q be a quiver and d is dimension vector $X = \operatorname{Rep}_d Q$. $G = GL_d(\mathbb{C}) = \prod_{d_i} GL_{d_i}(\mathbb{C})$.

then $X/\!/G$ depends on stability condition chosed. For example Q



then $X/\!/G \simeq \mathbb{P}^1$

3 <u>Kirwan-Ness theorem</u>, there are 2 methods to build "nice" quotient.

1) Via moment map. If X is projective and compact Lie group acts on X then

$$X/\!/G = \mu^{-1}(0)/G, \text{ where } \mu^*: X \to \mathfrak{g}^* \text{ – is moment map}$$

2) Or via GIT

$$X/\!/G = \operatorname{Proj} R^G.$$

There is fundamental Kirwan–Ness theorem wich connects this two quocient's.

<u>Teorem</u> Let K act on the graded ring $R = \mathbb{C}[x_1 \dots x_n]/I$ So $\operatorname{Proj} R$ is a variety of $\mathbb{CP}^{n-1}$, and (if smooth) a Hamiltonian K-manifold. Then there is identification

$$\mu^{-1}(0)/K \simeq (\operatorname{Proj} R)/K^{\mathbb{C}}.$$

where $K^{\mathbb{C}}$ is a complexification of K.

<u>Example</u> 7. Assume that $X = \mathbb{C}^{n \times n}$ and $K = U(n)$. Then $K^{\mathbb{C}} = GL(n)$. and $u^*(n)$ – is the set of self-adjoint matrixes.

~~Then~~

<u>Example</u> 8 Assume that $X = \mathbb{C}^n$, $K = S^1$, $G = K^{\mathbb{C}} = \mathbb{C}^*$. $\mu^{-1}(0)$ – is a ~~sph~~ spear S^{n-1} and K.-N. theorem says that

$$\mathbb{C}^n - \{0\}/\mathbb{C}^* \simeq S^{n-1}/S^1$$