Nakajima's quiver varieties

1) Recolectments on Hall algebras

Let Q - be a quiver. $Q = (Q_0, Q_1)$ (vertices, arrows).

Let C_Q - be a Cartan matrix of Q.

Let $\bar{Q}$ - be a double quiver, that is for each arrow $i \to j$ we associate one ~~more~~ arrow $j \to i$. Then $C_Q = 2I - A_{\bar{Q}}$, where $A_{\bar{Q}}$ - is ~~the~~ adjency matrix of $\bar{Q}$.

Define quantized Kac-Moody algebra ~~as~~ $U_v(g_Q)$ as an algebra over $\mathbb{C}(v)$. Given by ~~rela~~ generators $E_i, F_i, v^h, h \in \Pi$ subject to relations

$$v^h v^{h'} = v^{h+h'}, \quad h, h' \in \Pi$$

$$v^h E_j v^{-h} = v^{\alpha_j(h)} E_j, \quad j = 1, \dots, r$$

$$v^h F_j v^{-h} = v^{-\alpha_j(h)} F_j \quad j = 1, \dots r$$

$$[E_i, E_j] = \delta_{ij} \frac{v^{h_i} - v^{-h_i}}{v - v^{-1}}, \quad i, j = 1, \dots, r.$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \begin{bmatrix} 1-a_{ij} \\ \ell \end{bmatrix} E_i^\ell E_j E_i^{1-a_{ij}-\ell} = 0$$ } Quantum Serre relation,

where $[n] = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{-n} + v^{2-n} + \dots + v^{n-2} + v^n$.

$$\begin{bmatrix} t \\ n \end{bmatrix} = \frac{[t]!}{[r]!\,[t-r]!)} \quad [n]! = [n]\cdot[n-1]\cdots[1].$$

and Π is a vector space of ~~f~~ vector $\{h_1, \dots h_n\}$,
$\Pi^\vee$ is a dual $\{\alpha_1, \dots \alpha_n\}$ such that
$\alpha_i(h_j) = a_{ij}$ for all $i, j = 1, \dots r$.

For other side consider the representations of Q over $\mathbb{F}_q$ that is collections

$$\left((X_i)_{i\in Q_0},\ (x_\beta)_{\beta\in Q_1}: X_i \to X_j\right)$$

vector spaces — linear maps between the spaces.

$Rep(Q, \mathbb{F}_q)$ - be a category of finite dimensional representations of Q over $\mathbb{F}_q$.

Define the Hall algebra of $Rep(Q, \mathbb{F}_q)$. As vector space it has the basis of iso-classes of objects in $Rep(Q, \mathbb{F}_q)$.

$$\mathbb{H}(Q,q) \simeq \bigoplus_{M\in Ob(Rep(Q,\mathbb{F}_q))/\sim\, = X} \mathbb{C}[M].$$

with multiplication

$$[M]\cdot[N] = q^{\dim Hom(M,N) - \dim Ext(M,N)} \cdot \sum_{L\in X} \frac{P^L_{M,N}}{a_M a_N}\cdot[L]$$

$P^L_{M,N}$ - counts short exact sequences of the form

$0 \to N \to L \to M \to 0.$ $a_M = |Aut(M)|.$

Ringel showed

1) $\mathbb{H}(Q,q)$ is associative algebra.

2) Assignment $E_i \to [S_i]$, where S_i is a class of simple object concentrated at i. gives rise to homomorphism.

$$\beta: \mathcal{U}^+_v(\mathfrak{g}_Q) \to \mathbb{H}(Q,q), \text{ with } q^2 = v.$$

3) β is isomorphism $\Leftrightarrow$ Q is Dynkin Quiver.

③ Quiver varieties

Lustig used Ringel's ideas + perverse sheaves $\Rightarrow$ canonical basis of $U_q^+(g)$.

Nakajima aproach gives geometrical construction of whole $U(g)$, also the construction of $U_q(Lg)$ and simple integrable representations of $U(g)$ and $U_q(Lg)$.

In some sense his construction generalize the folowing constructions:

1) Iwahori – for Hecke algebra H_q
2) Beilinson–Lustig–MacPherson – for $U_v(sl_2)$.

The construction involves several steps:

1) We start with a quiver Q and associated two quivers to it $Q^\heartsuit$ and $\overline{Q^\heartsuit}$.
2) Construct "Hamiltonian reduction of Rep $\overline{Q^\heartsuit}$"
3) Construct Moduli space of $Rep\ Q^\heartsuit$ for fixed vectors (dimension) v & w namely, $\mathcal{M}_\Theta(v, w)$
4) Make ~~some top~~ Steinberg variety.

$$Z(w) = \prod_{(v,v') \in \mathbb{Z}^I \times \mathbb{Z}^I} \mathcal{M}_\Theta(v, w) \times \mathcal{M}_\Theta(v', w).$$

5) Make ~~algebra~~ space

$$H_w := \bigoplus_{m \geq 0} \left(\prod_{\substack{\{v,v'\} \in \mathbb{Z}^I \times \mathbb{Z}^I, \\ |v-v'| \leq m}} H_{[0]}(Z(w,v,v')) \right).$$

where $H_{[0]}$ is a homology space of $Z(w,v,v')$

6) Borel-Moore convolution in H_w extends to well defined operation on H_w which makes H_w an associative $\mathbb{C}$-algebra.

7). $\exists$ homomorphism between $\varphi: U(g_Q)$ and H_w.

8) the $U(g_Q)$ action on vector space L_w ~~gives~~ induced by φ gives a simple (integrable g_Q-module

Specifically one send's e_i, f_i, h_i to appropriated fundamental class in homology.

Quivers side (1), 2))

Q - be a quiver. $A_Q = (a_{ij})$ is the adjency matrix
a_{ij} - number of arrows between j and i

We have the scalar product on $\mathbb{C}^{Q_0}$ $(\cdot v, w)$.
Namely if $\alpha, \beta \in \mathbb{C}^{Q_0}$ then $\alpha \cdot \beta = \sum_{i \in Q_0} \alpha_i \beta_i$, thus
a symmetric form associated to A_Q is

$$A_Q \alpha \cdot \beta = \sum_{\rho \in Q_1} \alpha_i \beta_j$$

Let $\alpha \in \mathbb{C}^{Q_0}$, Conside the variety $R_\alpha = \bigoplus_{\rho \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j})$.

R_α - parametrize all the representations of Q with
dimension vector α.

$G_\alpha = \prod_{i \in Q_0} GL_{\alpha_i}(\mathbb{C})$, G_α acts on R_α via base change

$$(g_i)_{i \in Q_0} (X_\rho)_{i \in Q_0} \to (g_j \cdot X_\rho \cdot g_i^{-1})_{\rho: i \to j}$$

$$\left\{ \begin{array}{c} \text{iso classes of rep of } Q \\ \text{with dim vector } \alpha \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{orbits of the action} \\ \text{of } G_\alpha \text{ on } R_\alpha \end{array} \right\}.$$

~~Q^ρ is a β~~

$\dim R_\alpha = A_Q \alpha \cdot \alpha$, $\dim G_\alpha = \alpha \cdot \alpha$

So one can expect that the R_α / G_α should be a parametrizing space. Unfortunatelly it's rarely if having a variety X and algebraic group action on it the space X/G behaves well, (usually it's not Hausdorf.

For example If we will take $\mathbb{C}^n$ and $\mathbb{C}^*$ group acting by multiplication. then the orbit of 0

But deleting some badly behaved point's one can get a reasonable factor space.

For instance $(\mathbb{C}^n \setminus \{0\}) / \mathbb{C}^* - \mathbb{C}P^{n-1}$ - projective space

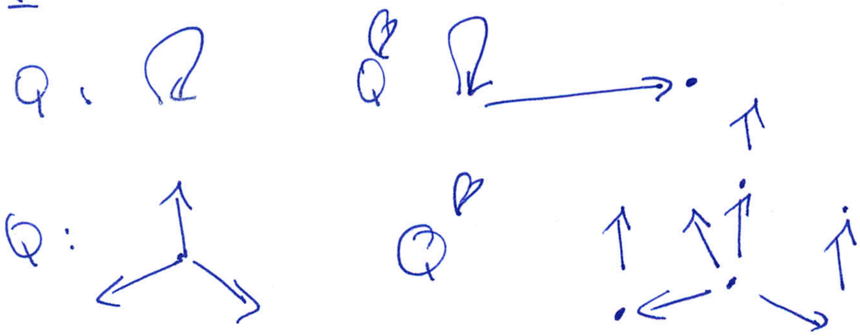
Such quotient usualy denoted by $X /\!/_\theta G$, where θ - is stability parameter which indicates, which points should be removed. (Mumford theory). (moduli space)

Doing the same for R_α we obtain the moduli space for quiver. Namely $R_\alpha /\!/_\theta G_\alpha = M_\theta(\alpha)$.

Framing

If we have a quiver Q. we make another quiver Q^f with $Q^f_0 = Q_0 \sqcup Q_0$ and $Q^f_1 = Q_1 \cup \{ i \to i' \mid i \in Q_0, i' \in Q_0' \}$.

Example



And we fix two dimensional vectors here one for Q, another one for

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Example

Q, Q^f

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