REALIZATIONS OF AFFINE LIE ALGEBRAS

1. Verma type modules

Let $\mathfrak{a}$ be a Lie algebra with a Cartan subalgebra H and root system Δ. A closed subset $P \subset \Delta$ is called a partition if $P \cap (-P) = \emptyset$ and $P \cup (-P) = \Delta$. If $\mathfrak{a}$ is finite-dimensional then every partition corresponds to a choice of positive roots in Δ and all partitions are conjugate by the Weyl group. The situation is different in the infinite-dimensional case. If $\mathfrak{a}$ is an affine Lie algebra then partitions are divided into a finite number of Weyl group orbits (cf. [JK], [F2]).

Given a partition P of Δ we define a *Borel* subalgebra $\mathfrak{b}_P \subset \mathfrak{a}$ generated by H and the root spaces $\mathfrak{a}_\alpha$ with $\alpha \in P$. All Borel subalgebras are conjugate in the finite-dimensional case. A *parabolic* subalgebra is a subalgebra that contains a Borel subalgebra. If $\mathfrak{p}$ is a parabolic subalgebra of a finite-dimensional $\mathfrak{a}$ then $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+$ where $\mathfrak{p}_0$ is a reductive Levi factor and $\mathfrak{p}_+$ is a nilpotent subalgebra. Parabolic subalgebras correspond to a choice of a basis π of the root system Δ and a subset $S \subset \pi$. A classification of all Borel subalgebras in the affine case was obtained in [F2]. In this case not all of them are conjugate but there exists a finite number of conjugacy classes. These conjugacy classes are parametrized by parabolic subalgebras of the underlined finite-dimensional Lie algebra. Namely, let $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+$ a parabolic subalgebra of $\mathfrak{G}$ containing a fixed Borel subalgebra $\mathfrak{b}$ of $\mathfrak{G}$. Define

$$B_{\mathfrak{p}} = \mathfrak{p}_+ \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{p}_0 \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

For any Borel subalgebra $\mathfrak{B}$ of $\tilde{\mathfrak{G}}$ there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{G}$ such that $\mathfrak{B}$ is conjugate to $B_{\mathfrak{p}}$.

When $\mathfrak{p}$ coincides with $\mathfrak{G}$, i.e. $\mathfrak{p}_+ = 0$, the corresponding Borel subalgebra $B_{\mathfrak{G}}$ is the *standard* Borel subalgebra defined by the choice of positive roots in $\tilde{\mathfrak{G}}$. Another extreme case is when $\mathfrak{p}_0 = \mathcal{H}$. This corresponds to the *natural* Borel subalgebra B_{nat} of $\tilde{\mathfrak{G}}$ considered in [JK].

Given a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{G}$ let $\lambda : B_{\mathfrak{p}} \to \mathbb{C}$ be a 1-dimensional representation of $B_{\mathfrak{p}}$. Then one defines an induced *Verma type* $\tilde{\mathfrak{G}}$-module

$$M_{\mathfrak{p}}(\lambda) = U(\tilde{\mathfrak{G}}) \otimes_{U(B_{\mathfrak{p}})} \mathbb{C}.$$

The module $M_{\mathfrak{G}}(\lambda)$ is the classical Verma module with highest weight λ [K]. In the case of natural Borel subalgebra we obtain *imaginary* Verma modules studied in [F1]. Note that the module $M_{\mathfrak{p}}(\lambda)$ is $U(\mathfrak{p}_-)$-free, where $\mathfrak{p}_-$ is the opposite subalgebra to $\mathfrak{p}_+$. The theory of Verma type modules was developed in [F2]. It follows immediately from the definition that, unless it is a classical Verma module, Verma type module with highest weight λ has a unique maximal submodule, it has both finite and infinite-dimensional weight spaces and it can be obtained using the parabolic induction from a standard Verma module M with highest weight λ over a certain affine Lie subalgebra. Moreover, if the central element c acts non-trivially on such Verma type module then the structure of this module is completely determined by the structure of module M, which is well-known ([F2], [C1]).

2. Boson type realizations of Verma modules

2.1. **Finite-dimensional case.** Consider first a finite-dimensional case. Let $\mathfrak{G} = LieG$ be a simple finite-dimensional Lie algebra with a Cartan decomposition $\mathfrak{G} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$. Take a Borel subalgebra $\mathfrak{b} = \mathfrak{n}_- \oplus \mathcal{H}$. Let $\mathfrak{b} = LieB$, $\mathfrak{n}_\pm = LieN_\pm$. Consider the flag variety $X = G/B$. Then X has a decomposition into open Schubert cells: $X = \cup_{w \in W} C(w)$, where $C(w) = B_+ w B_-/B_-$, $W = N(T)/T$ is the Weyl group and $T = B_+/N_+$. The codimension of $C(w)$ equals the length of w. The subgroup N_+ acts on X, and the largest orbit $\mathcal{U}$ of this action can be identified with proper N_+. From Section 3 we know that the Lie algebra $\mathfrak{G}$ can be mapped into vector fields on X and hence on $\mathcal{U}$. Thus $\mathfrak{G}$ can be embedded into the differential operators on $\mathcal{U}$ of degree ≤ 1. Since $N_+ \simeq \mathfrak{n}_+$ via the exponential map, we conclude that N_+ can be identified with an affine space $\mathbb{A}^{|\Delta_+|}$. Therefore, the ring of regular functions $\mathcal{O}_\mathcal{U}$ on $\mathcal{U}$ is just a polynomial ring in $m = |\Delta_+|$ variables and $\mathfrak{G}$ has an embedding into the Weyl algebra $\mathcal{A}_m$. In particular, $\mathcal{O}_\mathcal{U}$ is a $\mathfrak{G}$-module. In fact, a $\mathfrak{G}$-module $\mathcal{O}_\mathcal{U}$ is isomorphic to a contragradient module $M^*(0)$ with trivial highest weight.

Example 2.1. *Let $\mathfrak{G} = sl(2)$ with a standard basis e, f, h, $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Let $\mathfrak{b}_- = span\{f, h\}$. Then $G = SL_2(\mathbb{C})$ and the variety $X = G/B_-$ can be identified with the projective line $\mathbb{P}^1$ which has a big cell $\mathcal{U} = \mathbb{A}^1$. Denote $\mathcal{O}_\mathcal{U} = \mathbb{C}[x]$. Then one computes*

$$e \mapsto d/dx, h \mapsto -2xd/dx, f \mapsto -x^2 d/dx.$$

Hence, $M^*(0) \simeq \mathbb{C}[x]$ as a $\mathfrak{G}$-module (see Example 10.2.1 in [FZ]*).*

In order to obtain a geometrical realization of Verma modules one needs to consider the minimal 1-point orbit of N_+ on X.

Choosing another orbit of N_+ will give us a *twisted* Verma module. Twisted Verma modules are parametrized by the elements of the Weyl group and have the same character as corresponding Verma modules.

Remark 2.2. *Consider again example of $sl(2)$. Another way to get a realization of Verma module $M(0)$ on Fock space $\mathbb{C}[x]$ is the following. Apply to $sl(2)$ an automorphism which is a composition of two anti-involutions: $e \leftrightarrow f$, h is fixed, and $x \leftrightarrow d/dx$. Then it gives the following realization in second order differential operators: $f \mapsto x$, $h \mapsto -2xd/dx$, $e \mapsto -x(d/dx)^2$.*

2.2. **Affine case.** Consider now the loop algebra $\hat{\mathfrak{G}} = \mathfrak{G} \otimes \mathbb{C}[t, t^{-1}]$. Sometimes it is more convenient to consider a completion of $\hat{\mathfrak{G}}$ substituting the Laurent polynomials by the Laurent power series (for a geometric interpretation, but it is irrelevant to us. So jusy ignore the series). We will denote this Lie algebra by $\mathfrak{G}((t))$. The corresponding loop group will be denoted by $G((t))$. Fix a Cartan decomposition $\mathfrak{G} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$ and consider a Borel subalgebra $\mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathcal{H}$. Denote

$$\hat{\mathfrak{n}}_\pm = (\mathfrak{n}_\pm \otimes 1) \oplus (\mathfrak{G} \otimes t^\pm \mathbb{C}[[t^\pm]]),$$

$\hat{\mathfrak{b}}_\pm = \hat{\mathfrak{n}}_\pm \oplus \mathcal{H} \otimes \mathbb{C}[[t]]$. Let $\hat{N}_\pm$ and $\hat{B}_\pm$ be Lie groups corresponding to $\hat{\mathfrak{n}}_\pm$ and $\hat{\mathfrak{b}}_\pm$ respectively. Consider a flag variety $X = G((t))/\hat{B}_-$ which has a structure of a scheme of infinite type. As in the finite-dimensional case X splits into $\hat{N}_+$-orbits of finite codimension, parametrized by the affine Weyl group. There is an analogue of a big cell $\hat{\mathcal{U}}$ in X which is a projective limit of affine spaces, and hence, the ring of regular functions $\mathcal{O}_{\mathcal{U}}$ on $\mathcal{U}$ is a polynomial ring in infinitely many variables. Thus $\mathfrak{G}((t))$ acts on it by differential operators providing a realization for the contragradient Verma module with zero highest weight. Global sections of more general $\hat{N}_+$-equivariant sheaves on X will produce an arbitrary highest weight. Other $\hat{N}_+$-orbits in X correspond to twisted contragradient Verma modules. A striking difference with the finite-dimensional case is that we can not obtain standard Verma modules this way. They can be obtained considering $\hat{N}_+$-orbits on $G((t))/\hat{B}_+$.

3. First free field realization

In the previous section we considered the case of classical Verma modules for affine Lie algebras. Consider the completion $\mathfrak{b}_{nat}$ in $\mathfrak{G}((t))$ of the natural Borel subalgebra $\mathfrak{n}_- \otimes \mathbb{C}[t,t^{-1}] \oplus \mathcal{H} \otimes \mathbb{C}[t^{-1}]$. If N_- is the Lie group corresponding to $\mathfrak{n}_-$ then $\mathfrak{B}_{nat} = N_-((t))\mathcal{H}[[t^{-1}]]$ is the Borel subgroup corresponding to $\mathfrak{b}_{nat}$. Let $X = G((t))/\mathfrak{B}_{nat}$. The difference with the classical case is that X is not a scheme. This structure is called the *semi-infinite* manifold [FZ], [V1]. It can be viewed as the space of maps from $Spec\mathbb{C}((t))$ to the finite-dimensional flag variety G/B_-. We can consider the $\hat{N}_+$-orbits on the semi-infinite manifold and, in particular, $N_+((t))$ can be viewed as an analogue of the big cell $\mathcal{U}$ in G/B_-.

Example 3.1. *([FZ], 10.3.6). For simplicity we will only consider the case $\mathfrak{G} = sl(2)$. The corresponding semi-infinite manifold can be thought as $\mathbb{P}^1((t))$. The big cell $\mathbb{A}^1 \subset \mathbb{P}^1$ can be lifted to a big cell $\mathbb{A}^1((t)) = \{\begin{pmatrix} x(t) & -1\end{pmatrix}^t\}$, which coincides with the space of functions $F \simeq \mathbb{C}((t))$ on the punctured disc with the chosen coordinate t on the disc. Denote*

$$e_n = e \otimes t^n,\ h_n = h \otimes t^n,\ f_n = f \otimes t^n, n \in \mathbb{Z}.$$

Then the corresponding representation by vector fields on $\mathcal{F}$ is the following

$$e_n \mapsto \partial x_n,\ \ h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_m \partial x_{n+m},\ \ f_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_m x_k \partial x_{n+m+k}.$$

Let $V = \mathbb{C}[x_m, m \in \mathbb{Z}]$. It is clear that the differential operators corresponding to f_n are not well-defined on V (they take values in some formal completion of V). One way to deal with this problem is to apply the anti-involutions:

$$e_n \leftrightarrow f_n,\ \ h_n \leftrightarrow h_n;\ \ x_n \leftrightarrow \partial x_n,\ n \in \mathbb{Z}$$

which gives the following formulas:

$$f_n \mapsto \partial x_n,\ \ h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_{n+m}\partial x_m,\ \ e_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_{n+m+k}\partial x_m \partial x_k.$$

These formulas define the *first free field realization* of $\hat{sl}(2)$ in the polynomial ring $\mathbb{C}[x_m, m \in \mathbb{Z}]$. This module is, in fact, a quotient $M(0)$ of the imaginary Verma module with trivial highest weight by a submodule generated by the elements $h_n \otimes 1$, $n < 0$.

A boson type realization of the imaginary Verma module for $\hat{sl}(2)$ with a non-trivial central action was obtained by Bernard and Felder in the Fock space $\mathbb{C}[x_m, m \in \mathbb{Z}] \otimes \mathbb{C}[y_n, n > 0]$:

$$f_n \mapsto x_n, \;\; h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_{m+n}\partial x_m + \delta_{n<0}y_{-n} + \delta_{n>0}2nK\partial y_n + \delta_{n,0}J,$$

$$e_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_{k+m+n}\partial x_k \partial x_m + \sum_{k>0} y_k \partial x_{-k-n} + 2K\sum_{m>0} m\partial y_m \partial x_{m-n} + (Kn+J)\partial x_{-n}.$$

This module is irreducible if and only if $K \neq 0$. If we let $K = 0$ and quotient out the submodule generated by $y_m, m > 0$ then the factormodule is irreducible if and only if $J \neq 0$ (cf. [F1]). This construction has been generalized for all affine Lie algebras in [C2] providing a realization of imaginary Verma modules.

4. Second free field realization

There is another way to correct the formulas obtained in Example 6.1, which leads to the construction of Wakimoto modules [W].

Denote $a_n = \partial x_n$, $a_n^* = x_{-n}$ and consider formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \;\; a^*(z) = \sum_{n\in\mathbb{Z}} a_n^* z^{-n}.$$

Series $a(z)$ and $a^*(z)$ are called *formal distributions.* It is easy to see that $[a_n, a_m^*] = \delta_{n+m,0}$ and all other products are zero. The formulas in Example 6.1 can be rewritten as follows:

$$e(z) \mapsto a(z), \;\; h(z) \mapsto -2a^*(z)a(z), \;\; f(z) \mapsto -a^*(z)^2 a(z),$$

where $g(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n-1}$ for $g \in \{e, f, h\}$. This realization is not well-defined since the annihilation and creation operators are in a wrong order. It becomes well-defined after the application of two anti-involutions described above. Then the formulas read:

$$f(z) \mapsto a(z), \;\; h(z) \mapsto 2a(z)a^*(z), \;\; f(z) \mapsto -a(z)a^*(z)^2,$$

where a_n and a_n^* have the following meaning now $a_n = x_n$, $a_n^* = -\partial x_{-n}$. This is our quotient of the imaginary Verma module.

A different approach was suggested by Wakimoto ([W]) who introduced the *normal ordering.* Denote

$$a(z)_- = \sum_{n<0} a_n z^{-n-1}, \;\; a(z)+ = \sum_{n\geq 0} a_n z^{-n-1}$$

and define the normal ordering as follows

$$: a(z)b(z) := a(z)_- b(z) + b(z)a_+(z).$$

Let now

$$a_n = \left\{ \begin{array}{ll} x_n, & n<0 \\ \partial x_n, & n\geq 0, \end{array}\right. \quad a_n^* = \left\{ \begin{array}{ll} x_{-n}, & n\leq 0 \\ -\partial x_{-n}, & n>0, \end{array}\right. \quad b_m = \left\{ \begin{array}{ll} m\partial y_m, & m\geq 0 \\ y_{-m}, & m<0. \end{array}\right.$$

Here $[a_n, a_m^*] = [b_n, b_m] = \delta_{n+m,0}$.

Theorem 4.1. *([W]). The formulas*

$$c \mapsto K, \;\; e(z) \mapsto a(z), \;\; h(z) \mapsto -2 : a^*(z)a(z) : + b(z),$$

$$f(z) \mapsto - : a^*(z)^2 a(z) : + K\partial_z a^*(z) + a^*(z)b(z)$$

define the second free field realization *of the affine $sl(2)$ acting on the space* $\mathbb{C}[x_n, n \in \mathbb{Z}] \otimes \mathbb{C}[y_m, m > 0]$.

These modules are celebrated *Wakimoto modules.* They were defined for an arbitrary affine Lie algebra by Feigin and Frenkel [FF1], [FF2]. Generically Wkimoto modules are isomorphic to Verma modules.

5. Intermediate Wakimoto modules

So far we considered two extreme cases of Borel subalgebras in the affine Lie algebras: standard and natural. But if the rank of $\mathfrak{G}$ is more than 1 then the corresponding affine algebra has other conjugacy classes of Borel subalgebras. It should be possible to associate to each such Borel subalgebra a boson type realization by rearranging the annihilation and creation operators as it was done in the first and the second free field realizations. For affine Lie algebras of type $A_n^{(1)}$ associated with $sl(n+1)$ this has been accomplished in [CF], where a series of boson type realizations was constructed depending on the parameter $0 \leq r \leq n$. If $r = n$ this construction coincides with the construction of Wakimoto modules. On the other hand when $r = 0$ the obtained representation gives a Fock space realization described in [C2].

Let $0 \leq r \leq n$, $\gamma \in \mathbb{C}^*$, $k = \gamma^2 - (r+1)$. Let H_i, E_i, F_i, $i = 1, \dots, n$ be the standard basis for $\mathfrak{G} = sl(n+1)$. Denote $X_m = t^m \otimes X$ for $X, Y \in \mathfrak{G}$ and $m \in \mathbb{Z}$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for Δ^+, the positive set of roots for $\mathfrak{G}$, such that $H_i = \check{\alpha}_i$ and let Δ_r be the root system with basis $\{\alpha_1, \dots, \alpha_r\}$ ($\Delta_r = \emptyset$, if $r = 0$) of the Lie subalgebra $\mathfrak{G}_r = sl(r+1)$. Denote by $\mathcal{H}_r$ a Cartan subalgebra of $\mathfrak{G}_r$ spanned by H_i, $i = 1, \dots, r$. Set $\mathcal{H}_0 = 0$, $\tilde{\mathcal{H}_r} = \mathcal{H}_r \oplus \mathbb{C}c \oplus \mathbb{C}d$.

Denote by E_{im}, F_{im}, H_{im}, $i = 1, \dots, n$, $m \in \mathbb{Z}$, the generators of the loop algebra corresponding to $\mathfrak{G}$.

Let $\hat{\mathfrak{a}}$ be the infinite dimensional Heisenberg algebra with generators $a_{ij,m}$, $a^*_{ij,m}$, and $\mathbf{1}$, $1 \leq i \leq j \leq n$ and $m \in \mathbb{Z}$, subject to the relations

$$
\begin{aligned}
[a_{ij,m}, a_{kl,n}] &= [a^*_{ij,m}, a^*_{kl,n}] = 0, \\
[a_{ij,m}, a^*_{kl,n}] &= \delta_{ik}\delta_{jl}\delta_{m+n,0}\mathbf{1}, \\
[a_{ij,m}, \mathbf{1}] &= [a^*_{ij,m}, \mathbf{1}] = 0.
\end{aligned}
$$

This algebra acts on $\mathbb{C}[x_{ij,m} | i, j, m \in \mathbb{Z},\ 1 \leq i \leq j \leq n]$ by

$$
a_{ij,m} \mapsto \begin{cases} \partial/\partial x_{ij,m} & \text{if} \quad m \geq 0, \text{ and } j \leq r \\ x_{ij,m} & \text{otherwise,} \end{cases}
$$

$$
a^*_{ij,m} \mapsto \begin{cases} x_{ij,-m} & \text{if} \quad m \leq 0, \text{ and } j \leq r \\ -\partial/\partial x_{ij,-m} & \text{otherwise.} \end{cases}
$$

and $\mathbf{1}$ acts as an identity. Hence we have an $\hat{\mathfrak{a}}$-module generated by v such that

$$
a_{ij,m} v = 0, \quad m \geq 0 \text{ and } j \leq r, \quad a^*_{ij,m} v = 0, \quad m > 0 \text{ or } j > r.
$$

Let $\hat{\mathfrak{a}}_r$ denote the subalgebra generated by $a_{ij,m}$ and $a^*_{ij,m}$ and $\mathbf{1}$, where $1 \leq i \leq j \leq r$ and $m \in \mathbb{Z}$. If $r = 0$, we set $\hat{\mathfrak{a}}_r = 0$.

Let $((\alpha_i|\alpha_j))$ be the Cartan matrix for $sl(n+1)$ and let

$$
\mathfrak{B}_{ij} := (\alpha_i|\alpha_j)(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}).
$$

Let $\hat{\mathfrak{b}}$ be the Heisenberg Lie algebra with generators b_{im}, $1 \le i \le n$, $m \in \mathbb{Z}$, $\mathbf{1}$, and relations $[b_{im}, b_{jp}] = m\,\mathfrak{B}_{ij}\delta_{m+p,0}\mathbf{1}$ and $[b_{im}, \mathbf{1}] = 0$.

For each $1 \le i \le n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ acts on the space $\mathbb{C}[y_{i,m} | i, m \in \mathbb{N}^*,\ 1 \le i \le n]$ by

$$b_{i0} \mapsto \lambda_i, \quad b_{i,-m} \mapsto \mathbf{e}_i \cdot \mathbf{y}_m, \quad b_{im} \mapsto m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m} \quad \text{for} \quad m > 0$$

and $\mathbf{1} \mapsto 1$. Here

$$\mathbf{y}_m = (y_{1m}, \cdots, y_{nm}), \quad \frac{\partial}{\partial \mathbf{y}_m} = \left(\frac{\partial}{\partial y_{1m}}, \cdots, \frac{\partial}{\partial y_{nm}} \right)$$

and $\mathbf{e}_i$ are vectors in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$ where $\cdot$ means the usual dot product.

For any $1 \le i \le j \le n$, we define

$$a^*_{ij}(z) = \sum_{n \in \mathbb{Z}} a^*_{ij,n} z^{-n}, \quad a_{ij}(z) = \sum_{n \in \mathbb{Z}} a_{ij,n} z^{-n-1}$$

and

$$b_i(z) = \sum_{n \in \mathbb{Z}} b_{in} z^{-n-1}.$$

Then

$$[b_i(z), b_j(w)] = \mathfrak{B}_{ij}\partial_w \delta(z-w), \quad [a_{ij}(z), a^*_{kl}(w)] = \delta_{ik}\delta_{jl}\mathbf{1}\delta(z-w),$$

where

$$\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{w}\right)^n.$$

is the *formal delta function.*

Set

$$a_{ij}(z)_+ = a_{ij}(z), \quad a_{ij}(z)_- = 0$$
$$a^*_{ij}(z)_+ = 0, \quad a^*_{ij}(z)_- = a^*_{ij}(z),$$

if $j > r$.

Denote $\mathbb{C}[\mathbf{x}] = C[x_{ij,m} | i, j, m \in \mathbb{Z},\ 1 \le i \le j \le n]$ and $C[\mathbf{y}] = C[y_{i,m} | i, m \in \mathbb{N}^*,\ 1 \le i \le n]$.

Remark 5.1. *Note that*

$$: a_{ij}(z) a^*_{kl}(z) := \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} : a_{ij,n} a^*_{kl,m-n} : \right) z^{-m-1}$$

is well defined on $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])[[z, z^{-1}]]$ *for all* $l > r$ *or if* $l \le r$ *and* $j \le r$.

Let $\mathfrak{b}_r$ be the Borel subalgebra of $\tilde{\mathfrak{G}}$ corresponding to a parabolic subalgebra of $\mathfrak{G}$ whose semisimple part of the Levi factor is $\mathfrak{G}_r$.

Fix $\tilde{\lambda} \in \tilde{\mathcal{H}}^*$ and let $M_r(\tilde{\lambda})$ be the Verma type module associated with $\mathfrak{b}_r$ and $\tilde{\lambda}$. When $r = n$ this module is a standard Verma module while in the case $r = 0$ we get an imaginary Verma module. Denote by $v_{\tilde{\lambda}}$ the generator of $M_r(\tilde{\lambda})$.

Let $\tilde{\lambda}_r = \tilde{\lambda}|_{\tilde{\mathcal{H}}_r}$. The module $M_r(\tilde{\lambda})$ contains a $\tilde{\mathfrak{G}}_r$-submodule $M(\tilde{\lambda}_r) = U(\tilde{\mathfrak{G}}_r)(v_{\tilde{\lambda}})$ which is isomorphic to the standard Verma module for $\tilde{\mathfrak{G}}_r$. If $\tilde{\lambda}(c) \ne 0$ then the submodule structure of $M_r(\tilde{\lambda})$ is completely determined by the submodule structure of $M(\tilde{\lambda}_r)$ [C1], [FS].

Define

$$E_i(z) = \sum_{n\in\mathbb{Z}} E_{in} z^{-n-1}, \quad F_i(z) = \sum_{n\in\mathbb{Z}} F_{in} z^{-n-1}, \quad H_i(z) = \sum_{n\in\mathbb{Z}} H_{in} z^{-n-1},$$

$1 \le i \le n$.

Theorem 5.2. *([CF]). Let $\lambda \in \mathfrak{H}^*$ and set $\lambda_i = \lambda(H_i)$. The generating functions*

$$F_i(z) \mapsto a_{ii} + \textstyle\sum_{j=i+1}^n a_{ij} a^*_{i+1,j},$$

$$\begin{aligned} H_i(z) \mapsto 2 : a_{ii} a^*_{ii} : &+ \textstyle\sum_{j=1}^{i-1} \left(: a_{ji} a^*_{ji} : - : a_{j,i-1} a^*_{j,i-1} : \right) \\ &+ \textstyle\sum_{j=i+1}^n \left(: a_{ij} a^*_{ij} : - : a_{i+1,j} a^*_{i+1,j} : \right) + b_i, \end{aligned}$$

$$\begin{aligned} E_i(z) \mapsto : a^*_{ii} &\left(\textstyle\sum_{k=1}^{i-1} a_{k,i-1} a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki} a^*_{ki} \right) : + \textstyle\sum_{k=i+1}^n a_{i+1,k} a^*_{ik} \\ &- \textstyle\sum_{k=1}^{i-1} a_{k,i-1} a^*_{ki} - a^*_{ii} b_i - \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2 \right) \partial a^*_{ii}, \end{aligned}$$

$$c \mapsto \gamma^2 - (r+1)$$

define a representation on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$. In the above a_{ij}, a^*_{ij} and b_i denotes $a_{ij}(z)$, $a^*_{ij}(z)$ and $b_i(z)$ respectively.

This boson type realization of $\tilde{sl}(n+1)$ depends on the parameter $0 \le r \le n$ and defines a module structure on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ which is called an *intermediate Wakimoto module.* Denote it by $W_{n,r}(\lambda, \gamma)$ and consider a $\tilde{\mathfrak{G}}_r$-submodule $W = U(\tilde{\mathfrak{G}}_r)(v_{\tilde{\lambda}}) \simeq W_{r,r}(\lambda, \gamma)$ of $W_{n,r}(\lambda, \gamma)$. Then W is isomorphic to the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ ([FF2]), where $\lambda(r) = \lambda|_{\mathcal{H}_r}$, $\tilde{\gamma} = \gamma^2 - (r+1)$. Since generically Wakimoto modules are isomorphic to Verma modules, intermediate Wakimoto modules provide a realization for generic Verma type modules.

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