Lectures on Vertex Algebras

Why vertex algebras:

- powerful computational tool in rep. theory of inf. dim Lie algebras
- Monstrous Moonshine: connection between the Monster group and the modular function j
- Zhu's Theorem on modular invariance of the space of characters of rational VOAs
- Important part of string theory (CFT): describes correlation functions

1. Basic module for affine Lie algebra $\hat{sl}_2$

$$\hat{sl}_2 = sl_2 \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad K \text{ - central}$$

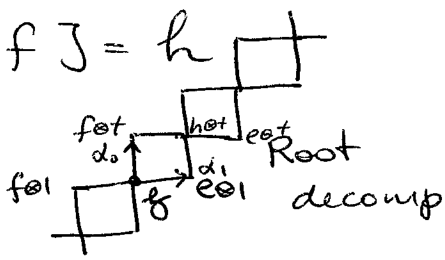
$$[xt^n, yt^s] = [x,y]t^{n+s} + n\delta_{n,-s}(x|y)K$$

$$[d, xt^n] = nxt^n$$

Basis of sl_2: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$$

$$(e|f) = 1, \quad (h|h) = 2$$



Cartan subalgebra of $\hat{sl}_2$: $\mathfrak{h} = \langle h\otimes 1, K, d \rangle$

Def of a representation / module

Basic module $L(\Lambda)$ - irreducible highest weight module

Lepowsky - Wilson 1981

Kac - Frenkel

Weight decomposition

$$L(\Lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} L(\Lambda)_\mu$$

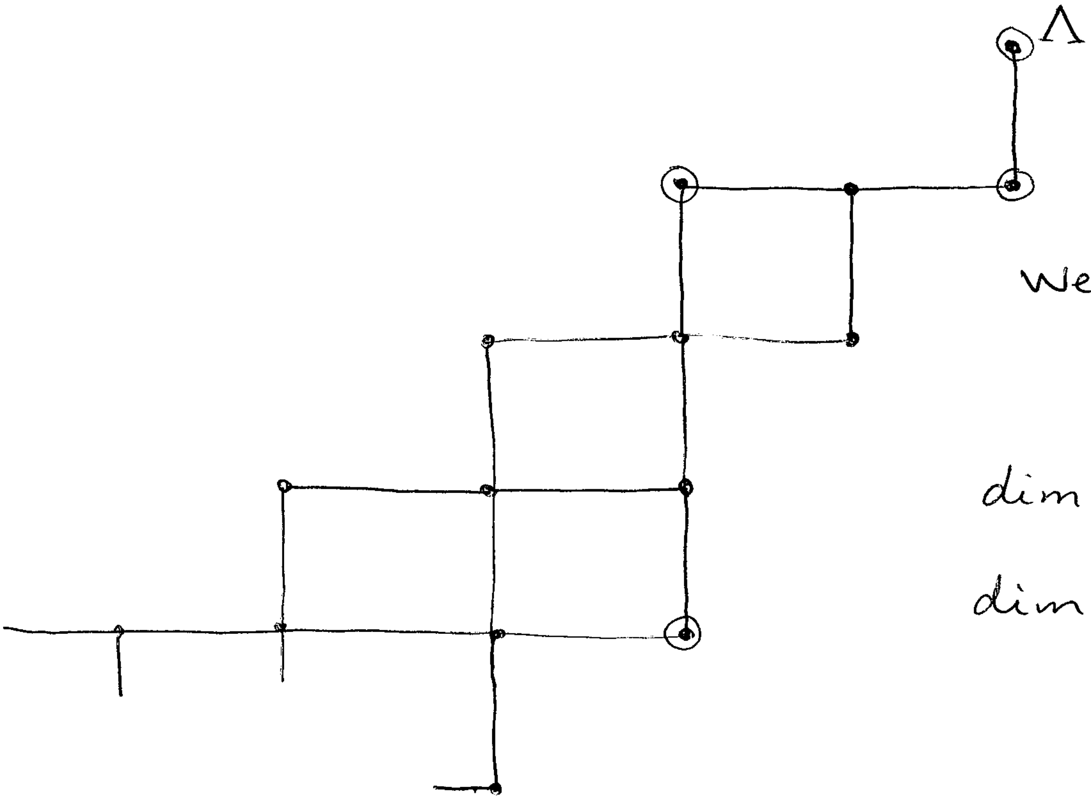
$\Lambda(h \otimes 1) = 0$

$\Lambda(K) = 1$

$\Lambda(d) = 0$ (irrelevant)

$$= \bigoplus_{n_0, n_1 \in \mathbb{Z}_+} L(\Lambda)_{\Lambda - n_0\alpha_0 - n_1\alpha_1}$$

$\rho(K) = \mathrm{Id}$



Weyl group action

$\dim L(\Lambda) = \infty$

$\dim L(\Lambda)_\mu < \infty$

$$\operatorname{char} L(\Lambda) = \sum_{\mu \in \mathfrak{h}^*} \dim L(\Lambda)_\mu \cdot e^\mu$$

$$= e^\Lambda \sum_{n_0, n_1 \in \mathbb{Z}_+} \dim L(\Lambda)_{\Lambda - n_0\alpha_0 - n_1\alpha_1} e^{-n_0\alpha_0 - n_1\alpha_1}$$

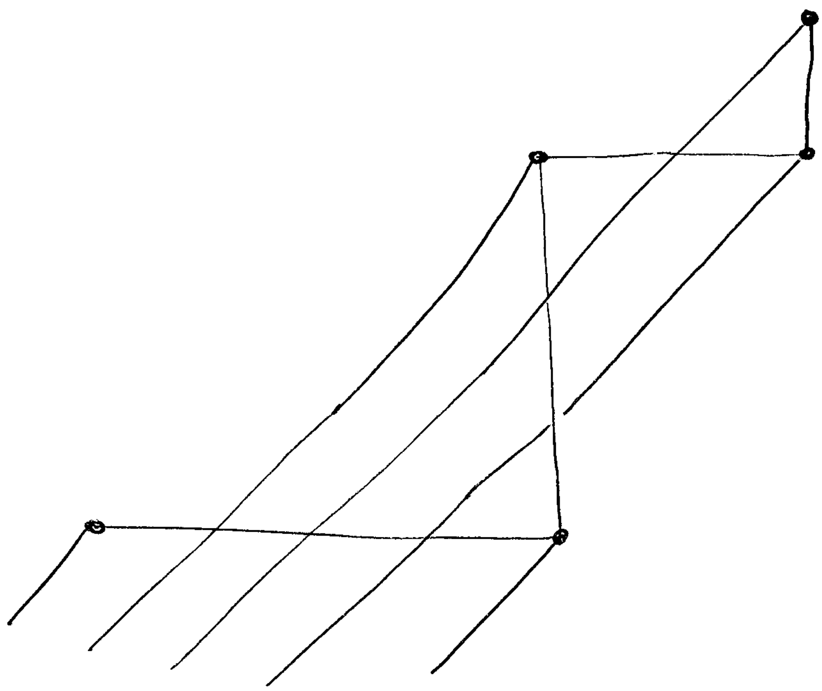
Kac-Weyl character formula:

$$\operatorname{char} L(\Lambda) = \sum_{w \in W} (-1)^w e^{w(\Lambda+\rho)-\rho} \times$$

$$\times \prod_{m \geq 1} (1 - e^{-m\alpha_0 - m\alpha_1})^{-1} (1 - e^{-(m-1)\alpha_0 - m\alpha_1})^{-1} \times (1 - e^{-m\alpha_0 - (m-1)\alpha_1})^{-1}$$

$= \ldots$ some non-trivial combinatorics ...

$$= \sum_{w \in \frac{W}{2}} e^{w(\Lambda)} \times \prod_{m \geq 1} (1 - e^{-m\alpha_0 - m\alpha_1})^{-1}$$



$\hat{sl}_2$ has an infinite-dim. Heisenberg subalgebra

$$h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \qquad [ht^n, ht^s] = 2n\,\delta_{n,-s} \cdot K$$

All irreducible highest weight modules with $\rho(K) \neq 0$ have the same structure and their characters are

$$e^{\mu} \cdot \prod_{m \geq 1} (1 - e^{-m\delta})^{-1}, \qquad \delta = \alpha_0 + \alpha_1$$

More explicitly, such a module is isomorphic to $\mathbb{C}[x_1, x_2, x_3, \dots]$

$$\rho(ht^{-n}) = n x_n$$
$$\rho(ht^{n}) = 2\frac{\partial}{\partial x_n}$$
$$\rho(K) = Id$$
$$\rho(h\otimes 1) = \alpha \cdot Id$$

Idea: Coordinatize $L(\Lambda)$:

$$L(\Lambda) = \mathbb{C}[q, q^{-1}] \otimes \mathbb{C}[x_1, x_2, x_3, \dots]$$

$$\mathrm{char}(q^n) = \Lambda - K^2\alpha_0 - K(K-1)\alpha_1$$
$$\rho(h)q^n = 2n\, q^n \Rightarrow \rho(h) = 2q\frac{\partial}{\partial q}$$

We know how Heisenberg acts on $L(\Lambda)$

How to obtain the action of the rest of the $\hat{sl}_2$?

Introduce formal power series

$$e(z) = \sum_{j\in\mathbb{Z}} et^j \cdot z^{-j-1}, \quad f(z) = \sum_{j\in\mathbb{Z}} ft^j \cdot z^{-j-1}$$

$$[ht^n, e(z)] = \sum_{j\in\mathbb{Z}} [ht^n, et^j] z^{-j-1}$$
$$= \sum_{j\in\mathbb{Z}} 2e t^{j+n} \cdot z^{-j-1} = 2z^n \sum_j et^{j+n} \cdot z^{-j-n-1}$$
$$= 2z^n e(z)$$

$$[ht^n, f(z)] = -2z^n f(z)$$

} achieve diagonalization

Want to determine $E(z) = \rho(e(z))$

Think that $E(z)$ is a differential operator that is made up of operators of differentiations and multiplications by x_i.

$$\left[2\frac{\partial}{\partial x_n}, E(z)\right] = 2z^n E(z)$$

$\frac{\partial}{\partial x_n}$ commutes with x_k, $k \neq n$ and all $\frac{\partial}{\partial x_j}$

this shows how $E(z)$ depends on x_n

$$\left.\begin{array}{l}\left[\frac{\partial}{\partial x}, f(x)\right] = f'(x) \\ \left[\frac{\partial}{\partial x}, f(x)\right] = \lambda f(x)\end{array}\right\} \Rightarrow f(x) = C\exp(\lambda x)$$

$$\Rightarrow E(z) \sim \exp(x_n z^n)$$

$$\Rightarrow E(z) \sim \exp\left(\sum_{n=1}^{\infty} x_n z^n\right)$$

Likewise $\left[n x_n, E(z)\right] = 2z^{-n} E(z)$

shows how $E(z)$ depends on $\frac{\partial}{\partial x_n}$

$$E(z) \sim \exp\left(-\sum_{n=1}^{\infty} \frac{2z^{-n}}{n}\frac{\partial}{\partial x_n}\right)$$

$$\Rightarrow E(z) \sim q\exp\left(\sum_{n=1}^{\infty} x_n z^n\right)\exp\left(-\sum_{n=1}^{\infty} \frac{2z^{-n}}{n}\frac{\partial}{\partial x_n}\right)$$

For example,

$$E(z)\cdot 1 = q\exp\left(\sum_{n=1}^{\infty} x_n z^n\right) =$$

$$= q\cdot z^0 + q x_1\cdot z + q\left(x_2 + \frac{x_1^2}{2}\right) z^2 + \dots$$

$\rho(et^{-1})\cdot 1 = q$ $\qquad$ $\rho(et^n)\cdot 1 = 0$ for $n \geq 0$

$\rho(et^{-2})\cdot 1 = q x_1$

$\rho(et^{-3})\cdot 1 = q\left(x_2 + \frac{x_1^2}{2}\right)$ etc.

Does not quite work when applied to other powers of q:

should be $\rho(e\otimes 1)q^{-1} \neq 0$

~~$\rho(e\otimes t)q^{-1} \neq 0$~~

$\rho(et)q^{-1} \neq 0$

Correct formula:

$$E(z) = q\, z^{2q\frac{\partial}{\partial q}}\cdot \exp\left(\sum_{n=1}^{\infty} x_n z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{2z^{-n}}{n}\frac{\partial}{\partial x_n}\right)$$

$$F(z) = q^{-1} z^{-2q\frac{\partial}{\partial q}} \exp\left(-\sum_{n=1}^{\infty} x_n z^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{2z^{-n}}{n}\frac{\partial}{\partial x_n}\right)$$

= vertex operator =

By construction, we know that ~~correct~~ we have correct Lie brackets ~~between~~

$[\rho(ht^n), \rho(et^s)] = 2\rho(et^{n+s})$

$[\rho(ht^n), \rho(ft^s)] = \dots$

We need still to check

$$[\rho(et^n), \rho(ft^s)] = 2\rho(ht^{n+s}) + n\delta_{n,-s}\cdot\rho(K)$$

$$[\rho(et^n), \rho(et^s)] = 0$$

$$[e(z_1), f(z_2)] = \sum_{i,j\in\mathbb{Z}} [et^i, ft^j]\, z_1^{-i-1} z_2^{-j-1}$$

$$= \sum_{ij} ht^{i+j} z_1^{-i-1} z_2^{-j-1} + \sum_i K i\, z_1^{-i-1} z_2^{+i-1}$$

$$= \sum_{i+j} ht^{i+j} z_2^{-i-j-1} \times \sum_i z_1^{-i-1} z_2^{i} + K\frac{\partial}{\partial z_2}\sum_i z_1^{-i-1} z_2^{i}$$

$$= h(z_2)\cdot z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + K\cdot z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)$$

δ-function: $\delta(z) = \sum_{j\in\mathbb{Z}} z^j$

Properties of δ-function

$$z^n\,\delta(z) = \delta(z)$$

$$P(z)\,\delta(z) = P(1)\,\delta(z)$$

$$\frac{\partial}{\partial z}: \quad P'(z)\delta(z) + P(z)\,\delta'(z) = P(1)\,\delta'(z)$$

$$\Rightarrow \quad P(z)\,\delta'(z) = P(1)\,\delta'(z) - P'(1)\,\delta(z)$$

$$\left(\frac{z_2}{z_1}\right)^{n}\delta\left(\frac{z_2}{z_1}\right) = \delta\left(\frac{z_2}{z_1}\right)$$

$$z_2\,\delta\left(\frac{z_2}{z_1}\right) = z_1\,\delta\left(\frac{z_2}{z_1}\right) \quad\Rightarrow\quad (z_2 - z_1)\,\delta\left(\frac{z_2}{z_1}\right) = 0$$

$$P(z_1, z_2)\,\delta\left(\frac{z_2}{z_1}\right) = P(z_2, z_2)\,\delta\left(\frac{z_2}{z_1}\right)$$

$$= P(z_1, z_1)\,\delta\left(\frac{z_2}{z_1}\right)$$

$$z_2 \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) = -z_1 \frac{\partial}{\partial z_1} \delta\left(\frac{z_2}{z_1}\right)$$

$$\frac{\partial P}{\partial z_1}(z_1, z_2)\, \delta\left(\frac{z_2}{z_1}\right) + P(z_1, z_2) \frac{\partial}{\partial z_1} \delta\left(\frac{z_2}{z_1}\right) = P(z_2, z_2) \frac{\partial}{\partial z_1} \delta\left(\frac{z_2}{z_1}\right)$$

$$\frac{\partial P}{\partial z_1}(z_2, z_2)\, \delta\left(\frac{z_2}{z_1}\right) - P(z_1, z_2) \cdot \left(\frac{z_2}{z_1}\right) \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right)$$

$$= - P(z_2, z_2) \left(\frac{z_2}{z_1}\right) \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right)$$

divide by z_2

$$\Rightarrow P(z_1, z_2)\; z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right)$$

$$= \frac{\partial P}{\partial z_1}(z_2, z_2)\; z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) + P(z_2, z_2)\; z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right)$$

$$[E(z_1), F(z_2)] = E(z_1) F(z_2) - F(z_2) E(z_1)$$

$$E(z_1) F(z_2) = q\, z_1^{2q\frac{\partial}{\partial q}} \exp\left(\sum_{n=1}^{\infty} x_n z_1^n\right) \exp\left(-\sum \frac{2 z_1^{-n}}{n} \frac{\partial}{\partial x_n}\right)$$

$$\times q^{-1} z_2^{-2q\frac{\partial}{\partial q}} \exp\left(-\sum_{n=1}^{\infty} x_n z_2^n\right) \exp\left(+\sum \frac{2 z_2^{-n}}{n} \frac{\partial}{\partial x_n}\right)$$

$$= \left(\frac{z_1}{z_2}\right)^{2q\frac{\partial}{\partial q}} \cdot z_1^{-2} \exp\left(\sum_n x_n (z_1^n - z_2^n)\right) \exp\left(-\sum \frac{2(z_1^{-n} - z_2^{-n})}{n} \frac{\partial}{\partial x_n}\right)$$

$$\times \exp\left(+\sum_{n=1}^{\infty} \frac{2}{n} \left(\frac{z_2}{z_1}\right)^n\right)$$

$$\exp\left(+2 \sum_{n=1}^{\infty} \frac{x^n}{n}\right) = \exp\left(-2 \ln(1-x)\right) = \frac{1}{(1-x)^2}$$

$$= \frac{d}{dx} \frac{1}{1-x} = 1 + 2x + 3x^2 + \ldots$$

$$z_1^{-2} \exp\left(\sum_{n=1}^{\infty} \frac{2}{n} \left(\frac{z_2}{z_1}\right)^n\right) = z_1^{-1} \left(\frac{1}{z_1} + 2 \frac{z_2}{z_1^2} + 3 \frac{z_2^2}{z_1^3} + \ldots\right)$$

Combining with $F(z_2)E(z_1)$ we get

$$\left(\frac{z_1}{z_2}\right)^{2q\frac{\partial}{\partial q}} \exp\left(\sum_{n=1}^{\infty} x_n (z_1^n - z_2^n)\right) \exp\left(-\sum_{n=1}^{\infty} \frac{2(z_1^{-n}-z_2^{-n})}{n}\frac{\partial}{\partial x_n}\right)$$

$$\times\ z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) =$$

$$= \frac{\partial P}{\partial z_1}(z_2, z_2) \cdot z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) + P(z_2, z_2)\, z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right)$$

$$P(z_2, z_2) = \mathrm{Id}$$

$$\frac{\partial P}{\partial z_1}(z_2, z_2) = 2q\frac{\partial}{\partial q} \cdot z_2^{-1} + \sum_{n=1}^{\infty} n x_n z_2^{+n-1}$$

$$+ \sum_{n=1}^{\infty} 2 \frac{\partial}{\partial x_n} z_2^{-n-1}$$

$$[E(z_1), F(z_2)] = H(z_2) \cdot z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) + \rho(K)\, z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right)$$

QED

Exercise $[E(z_1), E(z_2)] = 0$

Sugawara construction

Consider the Fock space $F = \mathbb{C}[x_1, x_2, x_3, \dots]$

with the action of the Heisenberg Lie algebra

$$h_n \mapsto \frac{\partial}{\partial x_n} \qquad h_0 \mapsto 0 \qquad [h_n, h_s] = n\delta_{n,-s}K$$

$$h_{-n} \mapsto n x_n \qquad K \mapsto \mathrm{id}$$

Form the Heisenberg field

$$h(z) = \sum_{j\in\mathbb{Z}} h_j z^{-j-1}$$

Exercise: $[h(z_1), h(z_2)] = K \cdot z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right)$

Note: while $h(z_1)h(z_2)$ is well-defined,

$h(z)\cdot h(z)$ leads to divergence:

$$\left(\sum_{n=1}^{\infty} n x_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\partial}{\partial x_n} z^{-n-1}\right)\left(\sum_{n=1}^{\infty} n x_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\partial}{\partial x_n} z^{-n-1}\right)$$

apply to 1 and compute the constant term of result:

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial x_n} z^{-n-1} \cdot n x_n z^{n-1} \cdot 1 = \sum_{n=1}^{\infty} n z^{-2} = \infty$$

There is a trick to avoid divergence:

- Normally ordered product

Take a field $a(z) = \sum_{j\in\mathbb{Z}} a_j z^{-j-1}$ and split it

into $\underbrace{a_+(z) = \sum_{j=-\infty}^{-1} a_j z^{-j-1}}_{\text{non-negative pow of } z}$ and $\underbrace{a_-(z) = \sum_{j=0}^{\infty} a_j z^{-j-1}}_{\text{neg pow of } z}$

Define

$$:a(z)\,b(z): \stackrel{def}{=} a_+(z)\,b(z) + b(z)\,a_-(z)$$

Example:

$$:h(z)h(z): = \left(\sum_{n=1}^{\infty} n x_n z^{n-1}\right) h(z)$$

$$+ h(z)\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial x_n} z^{-n-1}\right).$$ - well-defined

Sugawara construction:

Expand: $\frac{1}{2}:h(z)h(z): = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

<u>Exercise</u> Show that operators L_n on $\mathbb{C}[x_1, x_2, \ldots]$ satisfy the Virasoro algebra relations:

$$[L_n, L_s] = (n-s)L_{n+s} + \delta_{n,-s}\frac{n^3-n}{12}\cdot K.$$

Fields. Locality. Operations on fields.

Let V be a vector space. Typically $V = \bigoplus_{n=0}^{\infty} V_n$

A formal power series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, $a_n \in \operatorname{End} V$, is called a <u>field</u> if

$$\forall v\in V \quad \exists K\in\mathbb{Z} \quad \forall m\geq K \qquad a_m\cdot v = 0.$$

Typically $a_m V_n \subset V_{n+\alpha-m}$

Def. Two fields $a(z)$, $b(z)$ are called mutually local if

$$[a(z), b(w)] = \sum_{n=0}^{N} \frac{c^{[n]}(w)}{n!} z^{-1} \left(\frac{\partial}{\partial w}\right)^n \delta\left(\frac{w}{z}\right)$$

for some fields $c^{[0]}(w), \dots, c^{[N]}(w)$

Example: $E(z), F(z)$ were mutually local

$$[E(z), F(w)] = H(w)\, z^{-1} \delta\left(\frac{w}{z}\right) + K\, z^{-1} \frac{\partial}{\partial w} \delta\left(\frac{w}{z}\right)$$

Here $K = K \cdot w^0$

This is equivalent to: $(z-w)^{N+1} [a(z), b(w)] = 0$.

Note: It is non-trivial to say that $a(z)$ is ~~no~~ local with itself.

Exercise Let $\sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ be a field with components satisfying the Virasoro relations. Show that it is local with itself.

Operations on fields;

- $a(z) \longrightarrow \frac{\partial}{\partial z} a(z)$
- Normally ordered products

For local fields $a(z), b(z)$, extract the field $c^{[n]}(z)$ from the RHS of the commutator formula

Borcherds: All of this can be encoded in a unifying scheme:

n-th product of fields

$$a(w)_{(n)} b(w) \overset{\text{def}}{=}$$

$$\mathrm{Res}_z \left(a(z) b(w)\, i_w (z-w)^n - b(w) a(z)\, i_z (z-w)^n \right)$$

i_w - expansion in positive powers of w

i_z — " — — " — z

<u>Ex</u> $$i_w (z-w)^{-1} = i_w \frac{1}{z} \cdot \frac{1}{1-\frac{w}{z}} = z^{-1} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n$$

$$i_z (z-w)^{-1} = - z^{-1} \sum_{n=-\infty}^{-1} \left(\frac{w}{z}\right)^n$$

Note: for $n \geq 0$ $\quad i_w (z-w)^n = i_z (z-w)^n = (z-w)^n$

Let us compute $a(w)_{(n)} b(w)$ for $n \geq 0$:

$$\mathrm{Res}_z \left((z-w)^n \, [a(z), b(w)] \right)$$

$$= \mathrm{Res}_z \, (z-w)^n \sum_{k=0}^{N} \frac{c^{[k]}(w)}{k!} z^{-1} \left(\frac{\partial}{\partial w}\right)^k \delta\left(\frac{w}{z}\right)$$

For ~~k>n~~ $k < n$: $\quad (z-w)^n \left(\frac{\partial}{\partial w}\right)^k \delta\left(\frac{w}{z}\right) = 0$

For $k > n$, $\left(\frac{\partial}{\partial w}\right)^k \delta\left(\frac{w}{z}\right)$ does not have terms with $z^0, z^{-1}, \dots, z^{-k+1}$. However $(z-w)^n$ has $z^0, z^1, \dots, z^n$ so the residue is 0. Only the term $k=n$ contributes and we get $a(w)_{(n)} b(w) = c^{[n]}(w)$.

Let us take $n=-1$

$$a(w)_{(-1)} b(w) = Res_z \left(a(z)\, i_w (z-w)^{-1} \right) b(w)$$

$$- b(w)\, Res_z \left(a(z)\, i_z (z-w)^{-1} \right)$$

$$= Res_z \left(\sum_j a_j z^{-j-1} \times z^{-1} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n \right) b(w)$$

$$b(w) * Res_z \left(\sum_j a_j z^{-j-1} \times z^{-1} \sum_{n=-\infty}^{-1} \left(\frac{w}{z}\right)^n \right)$$

$$= \left(\sum_{j=-1}^{-\infty} a_j w^{-j-1} \right) b(w) + b(w) \sum_{j=0}^{\infty} a_j w^{-j-1}$$

$$= :a(w)\, b(w):$$

$$Res_z \left(a(z)\, i_w (z-w)^{-1} \right) = a(w)_+$$

Differentiating k times in w

$$Res \left(a(z)\, i_w (z-w)^{-k-1} \right) = \frac{1}{k!} \left(\frac{\partial}{\partial w} \right)^k a(w)_+$$

Thus

$$a(w)_{(-k-1)} b(w) = :\left(\frac{1}{k!} \left(\frac{\partial}{\partial w} \right)^k a(w) \right) b(w):$$

Dong's Lemma

Let the fields $a(z), b(z)$ be mutually local (including with themselves)

Then the fields $a(z), b(z), \frac{\partial}{\partial z} a(z)$, $a(z)_{(n)} b(z)$ are mutually local.

Definition of a vertex algebra

A vertex algebra is a vector space V together with the following structure:

$\mathbb{1} \in V$ identity element = vacuum

Map $D: V \to V$ (differentiation = infinitesimal translation)

State-field correspondence map

$$Y: V \longrightarrow \operatorname{End} V [[z, z^{-1}]]$$

For $v \in V$, write $Y(v, z) = \sum_{j \in \mathbb{Z}} v_{(j)} z^{-j-1}$

$$v_{(j)} \in \operatorname{End} V$$

satisfying axioms

Remark: In a usual algebra multiplication is a map $A \longrightarrow \operatorname{End} A$

$$a \mapsto L_a, \quad L_a(b) = a \cdot b.$$

so in a vertex algebra we have inf. many products

Axioms of a vertex algebra

1. $Y(\mathbb{1}, z) = \mathrm{Id} \cdot z^0$

2. $Y(a, z)\mathbb{1}$ contains no neg. powers of z

and $Y(a,z)\mathbb{1}\big|_{z=0} = a$ (self-replication)

3. $Y(Da, z) = [D, Y(a,z)] = \frac{\partial}{\partial z} Y(a,z)$

4. Locality $\forall a, b \in V \ \exists N \in \mathbb{N}$

$$(z-w)^{N+1} [Y(a,z), Y(b,w)] = 0$$

<u>Example</u> Let $\{a^{[\alpha]}(w)\}_{\alpha \in S}$ be a set of mutually local fields on a vector space U. Add the field $\mathbb{1} = \mathrm{Id} \cdot w^0$ and take the closure with respect to n-th products and derivatives, linear combinations. The resulting vector space V of fields is a vertex algebra.

$$D = \frac{\partial}{\partial w}$$

$$Y(a(w), z)\, b(w) = \sum_{n \in \mathbb{Z}} a(w)_{(n)} b(w) z^{-n-1}$$

Consequences of the axioms:

Commutator formula,

$$[Y(a,z), Y(b,w)] = \sum_{k=0}^{N} \frac{1}{k!} Y(a_{(k)}b, w)\, z^{-1} \left(\frac{\partial}{\partial w}\right)^k \delta\left(\frac{w}{z}\right)$$

in particular $a_{(k)}b = 0$ for $k \gg 0$

(so $Y(a,z)$ is a field on V)

More generally,

$$Y(a_{(n)}b, z) = Y(a,z)_{(n)} Y(b,z)$$

e.g. $Y(a_{(-1)}b, z) = :Y(a,z)\, Y(b,z):$

Let us revisit the basic module for $\hat{sl}_2$

$$\mathbb{C}[q,q^{-1}] \otimes \mathbb{C}[x_1, x_2, \dots]$$

Generalizing operators $E(z)$, $F(z)$, we construct vertex operators:

$$Y(q^n, z) = q^n z^{2nq\partial/\partial q} \exp\left(n \sum_{j=1}^{\infty} x_j z^j\right) \exp\left(-n \sum_{j=1}^{\infty} \frac{2z^{-j}}{j} \frac{\partial}{\partial x_j}\right)$$

We also have the field

$$H(z) = \sum_{j=1}^{\infty} j x_j z^{j-1} + 2q\frac{\partial}{\partial q} z^{-1} + \sum_{j=1}^{\infty} 2\frac{\partial}{\partial x_j} z^{-j-1}$$

$$H(z) = Y(\,?\,, z)$$

Use the self-replication axiom

$$a = Y(a,z)\mathbb{1}\Big|_{z=0} = H(z)\mathbb{1}\Big|_{z=0} = x_1$$

Then we get Note $(x_1)_{(-n)} = n x_n$

$$Y(x_1 q^k, z) = :Y(x_1, z) Y(q^k, z):$$

$$= :H(z) Y(q^k, z):$$

$$Y(x_{j_1} x_{j_2} \dots x_{j_s} q^k, z) =$$

$$= Y\left(\frac{1}{j_1}(x_1)_{(-j_1)} \dots \frac{1}{j_s}(x_1)_{(-j_s)} q^k, z\right) =$$

$$= \frac{1}{j_1! \dots j_s!} :\left(\frac{\partial}{\partial z}\right)^{j_1-1} H(z) \dots\dots :\left(\frac{\partial}{\partial z}\right)^{j_s-1} H(z)\, Y(q^k, z): \dots :$$

Existence Theorem (E. Frenkel - Kac - Radul - Wang)

Suppose we have $(V, \mathbb{1}, D)$

and a collection of fields $\{a^\alpha(z)\}_{\alpha \in S}$ on V

satisfying

(1) $[D, a^\alpha(z)] = \frac{\partial}{\partial z} a^\alpha(z)$

(2) $D\mathbb{1} = 0$, $a^\alpha(z)\mathbb{1} \in V[[z]]$

with $\{a^\alpha(z)\mathbb{1}\}_{\alpha \in S}$ - linearly independent

(3) $\{a^\alpha(z)\}_{\alpha \in S}$ is mutually local

(4) $V = \mathrm{Span}\left\{ a^{\alpha_1}_{(-j_1-1)} \cdots a^{\alpha_k}_{(-j_k-1)} \mathbb{1} \right\}$

Then V is a vertex algebra with Y map given by

$$Y\left(a^{\alpha_1}_{(-j_1-1)} \cdots a^{\alpha_k}_{(-j_k-1)} \mathbb{1}, z\right) =$$

$$= \frac{1}{j_1! \cdots j_k!} : \left(\frac{\partial}{\partial z}\right)^{j_1} a^{\alpha_1}(z) \cdots : \left(\left(\frac{\partial}{\partial z}\right)^{j_{k-1}} a^{\alpha_{k-1}}(z)\right) \left(\left(\frac{\partial}{\partial z}\right)^{j_k} a^{\alpha_k}(z)\right)$$

By existence theorem, the basic module is a vertex algebra.

This can be generalized to an arbitrary integral lattice

Two main sources of vertex algebras

- integral lattices
- infinite-dimensional Lie algebras

Vertex Lie algebras

Let $\mathcal{L}$ be a Lie algebra with a basis

$$\{ u(n),\ c(-1) \mid u \in \mathcal{U},\ c \in \mathcal{C} \}_{n \in \mathbb{Z}}$$

where $c(-1) \in Z(\mathcal{L})$.

Form the generating series

$$u(z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}$$

$$c(z) = c(-1) \cdot z^0$$

Let $\mathcal{F}$ be a subspace in $\mathcal{L}[[z, z^{-1}]]$ spanned by $\{u(z)$, $c(z)$ and their derivatives of all orders.

<u>Def</u>. $\mathcal{L}$ is a vertex Lie algebra if

$\forall a, b \in \mathcal{U} \quad \exists f^{[0]}(z), \ldots f^{[N]}(z) \in \mathcal{F}$

so that

$$[a(z), b(w)] = \sum_{k=0}^{N} \frac{f^{[k]}(w)}{k!} z^{-1} \left(\frac{\partial}{\partial w}\right)^k \delta\left(\frac{w}{z}\right).$$

Example: Affine Lie algebras

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

$\mathcal{U}$ = basis of $\mathfrak{g}$, $\quad \mathcal{C} = \{K\}$

$$x(n) = x \otimes t^n, \quad K = K(-1)$$

Verify that it is a vertex Lie algebra:

$$[x(z), y(w)] = \sum_{i,j \in \mathbb{Z}} [xt^i, yt^j] z^{-i-1} w^{-j-1}$$

$$= \sum_{ij} [x,y] t^{i+j} w^{-i-j-1} \cdot z^{-i-1} w^i$$

$$+ \sum_i i (x|y) \cdot K z^{-i-1} w^{i-1}$$

$$= [x,y](z) \cdot z^{-1} \delta\left(\frac{w}{z}\right) + (x|y) K(w) z^{-1} \frac{\partial}{\partial w} \delta\left(\frac{w}{z}\right).$$

Theorem (Dong-Li-Mason)

Let $\mathcal{L}$ be a vertex Lie algebra.

Construct the subspaces:

$\mathcal{L}_+ = \mathrm{Span}\{u(n) \mid n \geq 0, u \in \mathcal{U}\}$ - coeff at neg. pow of z

$\mathcal{L}_- = \mathrm{Span}\{u(n), c(-1) \mid n < 0, u \in \mathcal{U}, c \in \mathcal{C}\}$

Then

(1) Both $\mathcal{Z}_+$ and $\mathcal{Z}_-$ are subalgebras in $\mathcal{Z}$.

(2) Let $\mathbb{C}\mathbb{1}$ be a trivial 1-dim. repres. for $\mathcal{Z}_+$

Construct the induced module

$$V_{\mathcal{Z}} = \mathrm{Ind}_{\mathcal{Z}_+}^{\mathcal{Z}} \mathbb{C}\mathbb{1} \cong U(\mathcal{Z}_-) \otimes_{\mathbb{C}} \mathbb{1}.$$

$V_{\mathcal{Z}}$ has a structure of a vertex algebra with

$$Y(u_1(-j_1-1)\dots u_k(-j_k-1)\mathbb{1}, z) =$$

$$= \frac{1}{j_1!\dots j_k!} :\left(\frac{\partial}{\partial z}\right)^{j_1} u_1(z) \dots :\left(\left(\frac{\partial}{\partial z}\right)^{j_{k-1}} u_{k-1}(z)\right)\left(\frac{\partial}{\partial z}\right)^{j_k} u_k(z):\dots$$

(3) Let χ be a central character $\chi : \mathcal{C} \to \mathbb{C}$

Then $V_{\mathcal{Z}}(\chi) = V_{\mathcal{Z}} / \langle c(-1)\mathbb{1} - \chi(c)\mathbb{1} \rangle$ is a vertex algebra

(4) Let P be a maximal $\mathcal{Z}$-submodule in $V_{\mathcal{Z}}(\chi)$.

Then $L(\chi) = V_{\mathcal{Z}}(\chi)/P$ is a simple vertex algebra

(5) Any bounded module for $\mathcal{Z}$ is a module for the vertex algebra $V_{\mathcal{Z}}$.

Example Affine Lie algebras

$$\mathcal{L}_+ = \mathfrak{g} \otimes \mathbb{C}[t]$$

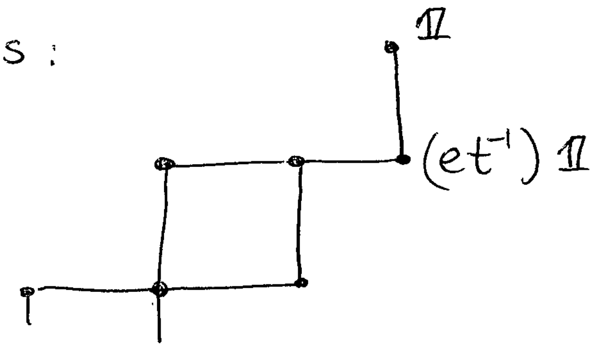
$$\mathcal{L}_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathbb{C}K$$

$$V_{\mathcal{L}} = U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathbb{C}K) \otimes \mathbb{1}$$

$$V_{\mathcal{L}}(c) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathbb{1}, \quad K \mapsto c \cdot \mathrm{id}$$

Let $c = 1$, $\mathfrak{g} = \mathfrak{sl}_2$ and consider the simple quotient $L(c)$. Then $L(c)$ is the basic module.

Applications:



In $L(1)$ $\quad (et^{-1})^2 \mathbb{1} = 0$

$$Y((et^{-1})^2 \mathbb{1}, z) = Y(e(-1)e(-1)\mathbb{1}, z)$$

$$= :e(z)e(z):$$

Then $:e(z)e(z): = 0$ on $L(1)$.

Virasoro vertex algebra

$$[L(n), L(s)] = (s-n)\, L(n+s) + \delta_{n,-s} \frac{n^3-n}{12} K$$

$$\mathcal{U} = \{\omega\}, \quad C = \{K\}$$

$$\omega(n) = L(n-1)$$

$$\omega(z) = \sum_{n\in\mathbb{Z}} \omega(n) z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

Exercise

$$[\omega(z), \omega(w)] = \frac{\partial}{\partial w}\omega(w)\, z^{-1}\delta\left(\frac{w}{z}\right)$$

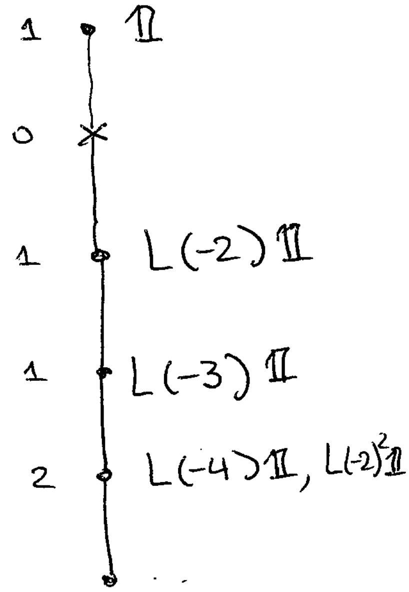
$$+ 2\omega(w)\, z^{-1}\frac{\partial}{\partial w}\delta\left(\frac{w}{z}\right)$$

$$+ \frac{1}{12} K(w)\, z^{-1}\left(\frac{\partial}{\partial w}\right)^3 \delta\left(\frac{w}{z}\right)$$

$$\mathcal{L}_- = \mathrm{Span}\{L(n) \mid n \leq -2, K\}$$

$$\mathcal{L}_+ = \mathrm{Span}\{L(n) \mid n \geq -1\}$$

$$V_{vir}(c) = U(\mathcal{L}_{\leq -2}) \otimes \mathbb{1}$$



Sugawara construction revisited

$\mathcal{H} = \langle h(n), K \rangle$

$$[h(n), h(s)] = n\delta_{n,-s} K$$

$$\mathcal{U} = \{h\}, \quad \mathcal{C} = \{K\}$$

$$[h(z), h(w)] = K(w) z^{-1} \frac{\partial}{\partial w} \delta\left(\frac{w}{z}\right).$$

$$\bar{V}_{Hei}(1) = \bar{U}(\mathcal{H}_{\leq -1}) \otimes \mathbb{1}$$

$$= \mathbb{C}[x_1, x_2, x_3, \dots]$$

$h(n) = \frac{\partial}{\partial x_n}$ $\qquad$ $H(z) = Y(x_1, z)$

$h(-n) = n x_n$

Sugawara: $\quad \omega(z) = \frac{1}{2} :H(z)H(z):$

is a Virasoro field.

Note: $\quad \omega = \frac{1}{2} x_1^2$

We need to prove

$$[\omega(z), \omega(w)] = \dots$$

Use the commutator formula

$$[Y(\omega, z), Y(\omega, w)] = \sum_{k=0}^{N} \frac{1}{k!} Y(\omega_{(k)}\omega, w) z^{-1} \frac{\partial^k}{\partial w} \delta$$

We need to compute $\omega_{(n)}\omega$ for $n \geq 0$

$n=0$: $\omega_{(0)}$ is z^{-1}-term in $\frac{1}{2}:H(z)H(z):$

apply to $\omega = \frac{1}{2}x_1^2$

$$\frac{1}{4}\left(\sum_{n=1}^{\infty} n x_n z^{n-1}\right)\left(\sum_{n=1}^{\infty} n x_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\partial}{\partial x_n} z^{-n-1}\right) x_1^2$$

$$+ \frac{1}{4}\left(\sum_{n=1}^{\infty} n x_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\partial}{\partial x_n} z^{-n-1}\right)\left(\sum_{n=1}^{\infty} \frac{\partial}{\partial x_n} z^{-n-1}\right) x_1^2$$

$n=0$: z^{-1} : $\omega_{(0)}\omega = \frac{1}{4} 2x_2 \frac{\partial}{\partial x_1} \cdot x_1^2 + \frac{1}{4} 2x_2 \frac{\partial}{\partial x_1} x_1^2$

$$= 2x_1 x_2 = h(-2)h(-1)\mathbb{1}.$$

$n=1$, z^{-2} : $\omega_{(1)}\omega = \frac{1}{4} x_1 \frac{\partial}{\partial x_1} \cdot x_1^2 + \frac{1}{4} x_1 \frac{\partial}{\partial x_1} \cdot x_1^2$

$$= x_1^2 = 2\omega$$

$n=2$ $\qquad \omega_{(2)}\omega = 0$

$n=3$ $\qquad \omega_{(3)}\omega = \frac{1}{4}\left(\frac{\partial}{\partial x_1}\right)^2 x_1^2 = \frac{1}{2}\mathbb{1}$

$\qquad \omega_{(n)}\omega = 0$ for $n > 3$.

$$[\omega(z), \omega(w)] = Y(\omega_{(0)}\omega, w)\, z^{-1}\delta\left(\frac{w}{z}\right)$$

$$+ Y(\omega_{(1)}\omega, w)\, z^{-1}\frac{\partial}{\partial w}\delta\left(\frac{w}{z}\right)$$

$$+ \frac{1}{3!} Y(\omega_{(3)}\omega, w)\, z^{-1}\left(\frac{\partial}{\partial w}\right)^3\delta\left(\frac{w}{z}\right)$$

$$Y(\omega_{(0)}\omega, z) = Y(h(-2)h(-1)\mathbb{1}, z)$$

$$= :\left(\frac{\partial}{\partial z} H(z)\right) H(z): = \frac{\partial}{\partial z} \omega(z)$$

$$[\omega(z), \omega(w)] = \frac{\partial}{\partial w} \omega(w) \; z^{-1} \delta\left(\frac{w}{z}\right)$$

$$+ 2\,\omega(w)\, z^{-1} \frac{\partial}{\partial w} \delta\left(\frac{w}{z}\right)$$

$$+ \frac{1}{12} K(w)\, z^{-1} \left(\frac{\partial}{\partial w}\right)^3 \delta\left(\frac{w}{z}\right).$$