OCCURRENCE IN PROBABILITY

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The goal of this note is to show how the notion of occurrence in probability for subsets of a product of a sequence of probability spaces can be used to provide a unified framework for transforming properties defined for sequences into their corresponding stochastic versions. The notions we consider are relative compactness of a sequence, the big O and the small o.

Big O and small o notation can be defined in a natural way for sequences of real numbers and the notion can be readily extended to sequences taking values in arbitrary normed spaces simply by replacing the elements of the sequence with their respective norms. For the theory we develop here the properties of a norm are actually irrelevant and we could replace normed spaces with sets endowed with some arbitrary nonnegative real-valued function that is interpreted as measuring the "size" of the elements of the set. Such function must be assumed measurable when we consider the stochastic version of big O and small o. Since we can't think of a situation in which such general set up is actually useful, we will consider only spaces endowed with semi-norms.

By a *semi-normed space* we mean a real vector space endowed with a semi-norm. All semi-normed spaces will be endowed with the semi-norm topology and all topological spaces will be endowed with the Borel σ-algebra. All semi-norms will be denoted by $\|\cdot\|$. By a *random object* we mean a measurable map defined on a probability space and taking values in a measurable space. The *distribution* of a random object X is defined as the probability measure on its counterdomain given by the push-forward under X of the probability measure on its domain.

We will always denote by $\big((\Omega_n, \mathcal{A}_n, P_n)\big)_{n\geq 1}$ a sequence of probability spaces and random objects carrying an index n will be assumed to be defined on Ω_n, unless otherwise explicitly stated.

Definition 1. Let $(\mathcal{X}_n)_{n\geq 1}$ and $(\mathcal{Y}_n)_{n\geq 1}$ be sequences of semi-normed spaces and, for each $n \geq 1$, let $x_n \in \mathcal{X}_n$ and $y_n \in \mathcal{Y}_n$ be given. We say that the sequence $(x_n)_{n\geq 1}$ is *large order* of the sequence $(y_n)_{n\geq 1}$, abbreviated as $x_n = O(y_n)$, if there exists $C \geq 0$ and $n_0 \geq 1$ such that $\|x_n\| \leq C\|y_n\|$ for all $n \geq n_0$. We say that the sequence $(x_n)_{n\geq 1}$ is *small order* of the sequence $(y_n)_{n\geq 1}$, abbreviated as $x_n = o(y_n)$, if for every $\eta > 0$ there exists $n_0 \geq 1$ such that $\|x_n\| \leq \eta\|y_n\|$ for all $n \geq n_0$.

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Definition 2. Let $(\mathcal{X}_n)_{n\geq 1}$ and $(\mathcal{Y}_n)_{n\geq 1}$ be sequences of semi-normed spaces and, for each $n \geq 1$, let X_n be an $\mathcal{X}_n$-valued random object and Y_n be a $\mathcal{Y}_n$-valued random object. We say that the sequence $(X_n)_{n\geq 1}$ is *stochastically of large order* of the sequence $(Y_n)_{n\geq 1}$, abbreviated as $X_n = O(Y_n)$, if for every $\varepsilon > 0$ there exists $C \geq 0$ and $n_0 \geq 1$ such that $P_n(\|X_n\| \leq C\|Y_n\|) \geq 1 - \varepsilon$, for all $n \geq n_0$. We say that the sequence $(X_n)_{n\geq 1}$ is *stochastically of small order* of the sequence $(Y_n)_{n\geq 1}$, abbreviated as $X_n = o(Y_n)$, if for every $\varepsilon > 0$ and every $\eta > 0$ there exists $n_0 \geq 1$ such that $P_n(\|X_n\| \leq \eta\|Y_n\|) \geq 1 - \varepsilon$, for all $n \geq n_0$.

Clearly $X_n = o(Y_n)$ if and only if for all $\eta > 0$ we have:

$$\lim_{n\to+\infty} P_n(\|X_n\| \leq \eta\|Y_n\|) = 1.$$

Remark 3. If $\|Y_n(\omega)\| = 0$ implies $\|X_n(\omega)\| = 0$ for all $\omega \in \Omega_n$ and all $n \geq 1$ then $\lim_{C\to+\infty} P_n(\|X_n\| \leq C\|Y_n\|) = 1$ for all $n \geq 1$ and therefore $X_n = O(Y_n)$ if and only if for all $\varepsilon > 0$ there exists $C \geq 0$ such that $P_n(\|X_n\| \leq C\|Y_n\|) \geq 1 - \varepsilon$ for all $n \geq 1$.

Remark 4. For each $n \geq 1$, let $(\Omega'_n, \mathcal{A}'_n, P'_n)$ be a probability space and $\phi_n : \Omega'_n \to \Omega_n$ be a measure-preserving map. Clearly $X_n = O(Y_n)$ (resp., $X_n = o(Y_n)$) if and only if $X_n \circ \phi_n = O(Y_n \circ \phi_n)$ (resp., $X_n \circ \phi_n = o(Y_n \circ \phi_n)$).

Remark 5. If the product $\mathcal{X}_n \times \mathcal{Y}_n$ of the semi-normed spaces $\mathcal{X}_n$ and $\mathcal{Y}_n$ is endowed with the product of the Borel σ-algebras of $\mathcal{X}_n$ and $\mathcal{Y}_n$ and with the probability measure given by the distribution of the random object $(X_n, Y_n) : \Omega_n \to \mathcal{X}_n \times \mathcal{Y}_n$ then Remark 4 applied to the measure-preserving maps $\phi_n = (X_n, Y_n)$ implies that $X_n = O(Y_n)$ (resp., $X_n = o(Y_n)$) if and only if $\pi^1_n = O(\pi^2_n)$ (resp., $\pi^1_n = o(\pi^2_n)$), where π^1_n and π^2_n denote respectively the first and the second projection of the product $\mathcal{X}_n \times \mathcal{Y}_n$. In particular, the conditions $X_n = O(Y_n)$ and $X_n = o(Y_n)$ depend only on the distribution of (X_n, Y_n) for all $n \geq 1$.

Definition 6. Let $\mathcal{X}$ be a topological space. A collection $\mathcal{P}$ of probability measures on $\mathcal{X}$ is called *tight* if for every $\varepsilon > 0$ there exists a compact Borel[1] subset K of $\mathcal{X}$ such that $P(K) \geq 1-\varepsilon$, for all $P \in \mathcal{P}$. We say that a sequence $(X_n)_{n\geq 1}$ of $\mathcal{X}$-valued random objects is *tight* if the collection consisting of the distributions of the random objects X_n is tight. More explicitly, $(X_n)_{n\geq 1}$ is tight if for every $\varepsilon > 0$ there exists a compact Borel subset K of $\mathcal{X}$ such that $P_n(X_n \in K) \geq 1 - \varepsilon$, for all $n \geq 1$.

Note that if $\mathcal{X}$ is a normed real finite-dimensional vector space then $(X_n)_{n\geq 1}$ is tight if and only if $X_n = O(1)$ (recall Remark 3).

Remark 7. As in Remark 4, if $\phi_n : \Omega'_n \to \Omega_n$ are measure-preserving maps then $(X_n)_{n\geq 1}$ is tight if and only if $(X_n \circ \phi_n)_{n\geq 1}$ is tight.

[1]Typically $\mathcal{X}$ is Hausdorff so that all compact sets are closed and hence Borel.

Remark 8. If $\mathcal{X}$ is a topological space, $(X_n)_{n\geq 1}$ is a sequence of $\mathcal{X}$-valued random objects and $\mathcal{X}_n$ denotes the probability space given by $\mathcal{X}$ endowed with its Borel σ-algebra and the distribution of X_n then the sequence $(X_n)_{n\geq 1}$ is tight if and only if the sequence of identity maps $\mathrm{Id} : \mathcal{X}_n \to \mathcal{X}$ is tight.

Remark 9. There are many relevant classes of topological spaces $\mathcal{X}$ such that every probability measure P on $\mathcal{X}$ is tight (meaning that the singleton $\{P\}$ is tight). This happens trivially, for instance, if $\mathcal{X}$ is a countable union of compact Borel subsets. A less trivial example consists of topological spaces that are homeomorphic to a Borel subset of a Polish space, i.e., a complete separable metric space ([1, Theorem 3.4.20]). Note that if the topological space $\mathcal{X}$ on which the random objects X_n take values is such that every probability measure on $\mathcal{X}$ is tight then $(X_n)_{n\geq 1}$ is tight if and only if for every $\varepsilon > 0$ there exists a compact Borel subset K of X and $n_0 \geq 1$ such that $P_n(X_n \in K) \geq 1 - \varepsilon$ for all $n \geq n_0$.

Definition 10. We say that a subset B of $\prod_{n=1}^{\infty} \Omega_n$ *occurs in probability* if for every $\varepsilon > 0$, there exists a sequence $(A_n)_{n\geq 1}$ with $A_n \in \mathcal{A}_n$ and $P_n(A_n) \geq 1 - \varepsilon$, for all $n \geq 1$, such that $\prod_{n=1}^{\infty} A_n \subset B$.

Proposition 11. *The collection of all subsets of $\prod_{n=1}^{\infty} \Omega_n$ that occur in probability is a σ-filter, i.e., it is nonempty, closed under countable intersections and every subset of $\prod_{n=1}^{\infty} \Omega_n$ that contains a subset in the collection is also in the collection.*

Proof. The only nontrivial statement is the fact that the collection is closed under countable intersections. Given a sequence $(B^k)_{k\geq 1}$ of subsets of $\prod_{n=1}^{\infty} \Omega_n$ that occur in probability and given $\varepsilon > 0$, pick for each $k \geq 1$ a sequence $(A_n^k)_{n\geq 1}$ with $A_n^k \in \mathcal{A}_n$ and $P_n(A_n^k) \geq 1 - \frac{\varepsilon}{2^k}$ for all $n \geq 1$ and such that $\prod_{n=1}^{\infty} A_n^k \subset B^k$. Setting $A_n = \bigcap_{k=1}^{\infty} A_n^k$ we then obtain $P_n(A_n) \geq 1 - \varepsilon$ for all $n \geq 1$ and $\prod_{n=1}^{\infty} A_n \subset \bigcap_{k=1}^{\infty} B^k$. □

Lemma 12. *Let $(\mathcal{X}_n)_{n\geq 1}$ and $(\mathcal{Y}_n)_{n\geq 1}$ be sequences of semi-normed spaces and for each $n \geq 1$, let X_n be an $\mathcal{X}_n$-valued random object and Y_n be a $\mathcal{Y}_n$-valued random object. Consider the following subsets of $\prod_{n=1}^{\infty} \Omega_n$:*

$$B_O = \left\{ (\omega_n)_{n\geq 1} \in \prod_{n=1}^{\infty} \Omega_n : X_n(\omega_n) = O\big(Y_n(\omega_n)\big) \right\},$$

$$B_o = \left\{ (\omega_n)_{n\geq 1} \in \prod_{n=1}^{\infty} \Omega_n : X_n(\omega_n) = o\big(Y_n(\omega_n)\big) \right\}.$$

We have that $X_n = O(Y_n)$ if and only if B_O occurs in probability and that $X_n = o(Y_n)$ if and only if B_o occurs in probability.

Proof. Assuming $X_n = O(Y_n)$, for any given $\varepsilon > 0$ pick $C \geq 0$ and $n_0 \geq 1$ as in the definition of $X_n = O(Y_n)$ and set $A_n = [\|X_n\| \leq C\|Y_n\|]$, for $n \geq n_0$, and $A_n = \Omega_n$ for $n < n_0$. Clearly $P_n(A_n) \geq 1 - \varepsilon$ for all n and $\prod_{n=1}^{\infty} A_n$

is contained in B_O. Conversely, assume that B_O occurs in probability and assume by contradiction that it is not true that $X_n = O(Y_n)$, so that there exists $\varepsilon > 0$ such that, for all $C \geq 0$, we have $P_n(\|X_n\| \leq C\|Y_n\|) < 1-\varepsilon$ for infinitely many n. We can then obtain a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that

$$P_{n_k}(\|X_{n_k}\| \leq k\|Y_{n_k}\|) < 1 - \varepsilon, \tag{1}$$

for all $k \geq 1$. If $(A_n)_{n\geq 1}$ is a sequence of nonempty sets as in the definition of occurrence in probability for B_O then (1) implies that A_{n_k} is not contained in $[\|X_{n_k}\| \leq k\|Y_{n_k}\|]$ and therefore we can obtain a sequence $(\omega_n)_{n\geq 1}$ in $\prod_{n=1}^{\infty} A_n$ such that $\|X_{n_k}(\omega_{n_k})\| > k\|Y_{n_k}(\omega_{n_k})\|$ for all $k \geq 1$. Clearly $(\omega_n)_{n\geq 1}$ is not in B_O, which contradicts $\prod_{n=1}^{\infty} A_n \subset B_O$.

Assume now that $X_n = o(Y_n)$ and let $\varepsilon > 0$ be fixed. We can obtain a sequence $(n_k)_{k\geq 1}$ of positive integers such that, for all $k \geq 1$, we have $P_n\big(\|X_n\| \leq \frac{1}{k}\|Y_n\|\big) \geq 1-\varepsilon$ for all $n \geq n_k$ and we can assume that $(n_k)_{k\geq 1}$ is strictly increasing. Setting $A_n = \Omega_n$ for $n < n_1$ and $A_n = \big[\|X_n\| \leq \frac{1}{k}\|Y_n\|\big]$ for $n_k \leq n < n_{k+1}$ and all $k \geq 1$, we have that $P_n(A_n) \geq 1 - \varepsilon$ for all $n \geq 1$ and that $\prod_{n=1}^{\infty} A_n$ is contained in B_o, proving that B_o occurs in probability. Conversely, assume that B_o occurs in probability and assume by contradiction that it is not true that $X_n = o(Y_n)$, so that there exists $\varepsilon > 0$ and $\eta > 0$ such that

$$P_n(\|X_n\| \leq \eta\|Y_n\|) < 1 - \varepsilon, \tag{2}$$

for infinitely many n. Let $(A_n)_{n\geq 1}$ be a sequence of nonempty sets as in the definition of occurrence in probability for B_o. For those n such that (2) holds, we have that A_n is not contained in $[\|X_n\| \leq \eta\|Y_n\|]$ and therefore we can obtain a sequence $(\omega_n)_{n\geq 1}$ in $\prod_{n=1}^{\infty} A_n$ such that $\|X_n(\omega_n)\| > \eta\|Y_n(\omega_n)\|$, for infinitely many n. This contradicts $(\omega_n)_{n\geq 1} \in B_o$ and concludes the proof. □

Remark 13. If we let $\mathcal{X}_n \times \mathcal{Y}_n$ be endowed with the product of the Borel σ-algebras of the semi-normed spaces $\mathcal{X}_n$ and $\mathcal{Y}_n$ and with the probability measure given by the distribution of (X_n, Y_n) then Remark 5 and Lemma 12 applied to the projections of $\mathcal{X}_n \times \mathcal{Y}_n$ imply that $X_n = O(Y_n)$ if and only if the set

$$\left\{ \big((x_n, y_n)\big)_{n\geq 1} \in \prod_{n=1}^{\infty} (\mathcal{X}_n \times \mathcal{Y}_n) : x_n = O(y_n) \right\}$$

occurs in probability and that $X_n = o(Y_n)$ if and only if the set

$$\left\{ \big((x_n, y_n)\big)_{n\geq 1} \in \prod_{n=1}^{\infty} (\mathcal{X}_n \times \mathcal{Y}_n) : x_n = o(y_n) \right\}$$

occurs in probability.

We recall that a measurable space is called *standard Borel* if it is isomorphic to a Borel subset of a Polish space endowed with its Borel σ-algebra.

Lemma 14. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space such that* $(\Omega, \mathcal{A})$ *is standard Borel and* $\phi : \Omega \to \mathcal{X}$ *be a measurable map taking values in a topological space* $\mathcal{X}$ *that is homeomorphic to a Borel subset of a Polish space. For every* $A \in \mathcal{A}$*, the following inequality holds:*

$$\sup \big\{ P\big(\phi^{-1}[K]\big) : K \subset \phi[A] \text{ compact} \big\} \geq P(A).$$

Proof. Let Q be the push-forward of P by ϕ and denote by $\overline{P}$ and $\overline{Q}$ the completions of the measures P and Q, respectively. Clearly ϕ is a measure-preserving map with respect to such completions. We have that $\phi[A]$ is analytic (inside some Polish space containing $\mathcal{X}$) and thus it belongs to the domain of $\overline{Q}$ ([1, Theorem 4.10.12]), so that it has the same probability as some Borel subset of $\mathcal{X}$ contained in $\phi[A]$. Since the probability of a Borel subset of a Polish space equals the supremum of the probabilities of its compact subsets ([1, Theorem 3.4.20]), we have:

$$\begin{aligned} \overline{Q}\big(\phi[A]\big) &= \sup \big\{ Q(K) : K \subset \phi[A] \text{ compact} \big\} \\ &= \sup \big\{ P\big(\phi^{-1}[K]\big) : K \subset \phi[A] \text{ compact} \big\}. \end{aligned}$$

Finally, the fact that ϕ is measure-preserving with respect to $\overline{P}$ and $\overline{Q}$ yields:

$$\overline{Q}\big(\phi[A]\big) = \overline{P}\big(\phi^{-1}[\phi[A]]\big) \geq P(A).$$

□

Lemma 15. *Let* $\mathcal{X}$ *be a topological space and* $(X_n)_{n\geq 1}$ *be a sequence of* $\mathcal{X}$*-valued random objects. Consider the subset* B *of* $\prod_{n=1}^{\infty} \Omega_n$ *consisting of those sequences* $(\omega_n)_{n\geq 1}$ *such that* $\{X_n(\omega_n) : n \geq 1\}$ *is contained in a compact subset of* $\mathcal{X}$*. If* $(X_n)_{n\geq 1}$ *is tight then* B *occurs in probability. The converse holds if* $\mathcal{X}$ *is homeomorphic to a Borel subset of a Polish space and all measurable spaces* $(\Omega_n, \mathcal{A}_n)$ *are standard Borel.*

Proof. If $(X_n)_{n\geq 1}$ is tight, given $\varepsilon > 0$ we pick a compact Borel subset K of $\mathcal{X}$ as in the definition of tight sequences and we set $A_n = [X_n \in K]$, for all $n \geq 1$. Clearly $P_n(A_n) \geq 1 - \varepsilon$ for all $n \geq 1$ and $\prod_{n=1}^{\infty} A_n \subset B$. To prove the converse, let $\varepsilon > 0$ be given and let $(A_n)_{n\geq 1}$ be a sequence of nonempty sets with $A_n \in \mathcal{A}_n$ and $P_n(A_n) > 1 - \varepsilon$, for all $n \geq 1$, and with $\prod_{n=1}^{\infty} A_n \subset B$. By Lemma 14 there exists a sequence $(K_n)_{n\geq 1}$ of compact subsets of $\mathcal{X}$ such that $K_n \subset X_n[A_n]$ and $P_n(X_n \in K_n) > 1 - \varepsilon$, for all $n \geq 1$. We will show that there exists a compact subset K of $\mathcal{X}$ containing $\bigcup_{n=1}^{\infty} K_n$ and this will conclude the proof that $(X_n)_{n\geq 1}$ is tight, as $P_n(X_n \in K) > 1 - \varepsilon$ for all $n \geq 1$. Since $\mathcal{X}$ is metrizable, to prove the existence of K it is sufficient to show that every sequence in $\bigcup_{n=1}^{\infty} K_n$ has a subsequence that is convergent in $\mathcal{X}$. If a sequence in $\bigcup_{n=1}^{\infty} K_n$ has infinitely many terms in K_n for some $n \geq 1$ then it has a convergent subsequence, because K_n is compact. If not, it contains a subsequence $(x_k)_{k\geq 1}$ such that $x_k \in K_{n_k}$ for all $k \geq 1$, where $(n_k)_{k\geq 1}$ is an injective sequence of positive integers. We can then find a sequence $(\omega_n)_{n\geq 1} \in \prod_{n=1}^{\infty} A_n$ such that $x_k = X_{n_k}(\omega_{n_k})$, for all $k \geq 1$. Since $(\omega_n)_{n\geq 1}$ is in B we obtain that $\{x_k : k \geq 1\}$ is contained in a compact subset of $\mathcal{X}$, concluding the proof. □

Remark 16. If $\mathcal{X}$ is a topological space homeomorphic to a Borel subset of a Polish space and $(X_n)_{n\geq 1}$ is a sequence of $\mathcal{X}$-valued random objects then it follows from Remark 8 and Lemma 15 that $(X_n)_{n\geq 1}$ is tight if and only if the set of sequences

$$\left\{(x_n)_{n\geq 1} \in \prod_{n=1}^{\infty} \mathcal{X}_n : \{x_n : n \geq 1\} \text{ is contained in a compact subset of } \mathcal{X}\right\}$$

occurs in probability, where $\mathcal{X}_n$ denotes the probability space given by $\mathcal{X}$ endowed with its Borel σ-algebra and the distribution of X_n.

Though the full precise statement of our next result is large and ugly, it's meaning is simple: suppose that we wish to establish some implication whose antecedent and consequent are statements that certain sequences of random objects are tight or that they are stochastically of large order or of small order of some other sequences of random objects. If the number of sequences involved is countable, our next result says that it is sufficient to establish the corresponding deterministic version of the implication, i.e., the implication involving sequences obtained by evaluating the random objects at particular points of the probability spaces.

Proposition 17. *Consider the following set of data:*

- *countable sets* Λ, Λ' *and families*

$$(\mathcal{X}_{\lambda,n})_{\lambda\in\Lambda,n\geq 1}, \quad (\mathcal{Y}_{\lambda,n})_{\lambda\in\Lambda,n\geq 1}, \quad (\mathcal{X}'_{\lambda,n})_{\lambda\in\Lambda',n\geq 1}, \quad (\mathcal{Y}'_{\lambda,n})_{\lambda\in\Lambda',n\geq 1}$$

  of semi-normed spaces;
- *families* $(X_{\lambda,n})_{\lambda\in\Lambda,n\geq 1}$, $(Y_{\lambda,n})_{\lambda\in\Lambda,n\geq 1}$, $(X'_{\lambda,n})_{\lambda\in\Lambda',n\geq 1}$, $(Y'_{\lambda,n})_{\lambda\in\Lambda',n\geq 1}$ *of random objects such that* $X_{\lambda,n}$ *takes values in* $\mathcal{X}_{\lambda,n}$, $Y_{\lambda,n}$ *takes values in* $\mathcal{Y}_{\lambda,n}$, $X'_{\lambda,n}$ *takes values in* $\mathcal{X}'_{\lambda,n}$ *and* $Y'_{\lambda,n}$ *takes values in* $\mathcal{Y}'_{\lambda,n}$;
- *a countable set* Γ, *a family* $(\mathcal{Z}_\gamma)_{\gamma\in\Gamma}$ *of topological spaces and a family of random objects* $(Z_{\gamma,n})_{\gamma\in\Gamma,n\geq 1}$ *with* $Z_{\gamma,n}$ *taking values in* $\mathcal{Z}_\gamma$;
- *sequences* $(\mathcal{X}''_n)_{n\geq 1}$, $(\mathcal{Y}''_n)_{n\geq 1}$ *of semi-normed spaces;*
- *sequences* $(X''_n)_{n\geq 1}$ *and* $(Y''_n)_{n\geq 1}$ *of random objects such that* X''_n *takes values in* $\mathcal{X}''_n$ *and* Y''_n *takes values in* $\mathcal{Y}''_n$.

Assume that $X_{\lambda,n} = O(Y_{\lambda,n})$ *for all* $\lambda \in \Lambda$, *that* $X'_{\lambda,n} = o(Y'_{\lambda,n})$ *for all* $\lambda \in \Lambda'$ *and that* $(Z_{\gamma,n})_{n\geq 1}$ *is tight for all* $\gamma \in \Gamma$. *Assume also that for every sequence* $(\omega_n)_{n\geq 1}$ *in* $\prod_{n=1}^{\infty} \Omega_n$ *the following condition holds:*

(*) *if* $X_{\lambda,n}(\omega_n) = O\big(Y_{\lambda,n}(\omega_n)\big)$ *for all* $\lambda \in \Lambda$, $X'_{\lambda,n}(\omega_n) = o\big(Y'_{\lambda,n}(\omega_n)\big)$ *for all* $\lambda \in \Lambda'$ *and the set* $\{Z_{\gamma,n}(\omega_n) : n \geq 1\}$ *is contained in a compact subset of* $\mathcal{Z}_\gamma$ *for all* $\gamma \in \Gamma$ *then* $X''_n(\omega_n) = O\big(Y''_n(\omega_n)\big)$ *(resp.,* $X''_n(\omega_n) = o\big(Y''_n(\omega_n)\big)$*).*

Under such conditions, we have that $X''_n = O(Y''_n)$ *(resp.,* $X''_n = o(Y''_n)$*).*

Proof. Denote by B the set of sequences $(\omega_n)_{n\ge1}$ in $\prod_{n=1}^{\infty}\Omega_n$ such that:

$$X_{\lambda,n}(\omega_n) = O\big(Y_{\lambda,n}(\omega_n)\big), \text{ for all } \lambda\in\Lambda,$$
$$X'_{\lambda,n}(\omega_n) = o\big(Y'_{\lambda,n}(\omega_n)\big) \text{ for all } \lambda\in\Lambda' \text{ and}$$
$$\{Z_{\gamma,n}(\omega_n) : n\ge1\} \text{ is contained in a compact subset of } \mathcal{Z}_\gamma \text{ for all } \gamma\in\Gamma.$$

Since Λ, Λ' and Γ are countable, the fact that $X_{\lambda,n} = O(Y_{\lambda,n})$ for all $\lambda\in\Lambda$, $X'_{\lambda,n} = o(Y'_{\lambda,n})$ for all $\lambda\in\Lambda'$ and $(Z_{\gamma,n})_{n\ge1}$ is tight for all $\gamma\in\Gamma$ imply, by Lemmas 12, 15 and Proposition 11, that B occurs in probability. As assumption (*) holds for every $(\omega_n)_{n\ge1}\in\prod_{n=1}^{\infty}\Omega_n$, we have that B is contained in the set of sequences $(\omega_n)_{n\ge1}\in\prod_{n=1}^{\infty}\Omega_n$ such that

$$X''_n(\omega_n) = O\big(Y''_n(\omega_n)\big) \text{ (resp., } X''_n(\omega_n) = o\big(Y''_n(\omega_n)\big)\text{)},$$

so that the latter set also occurs in probability. This yields $X''_n = O(Y''_n)$ (resp., $X''_n = o(Y''_n)$) by Lemma 12. □

Remark 18. In the statement of Proposition 17 it is clearly sufficient to assume that condition (*) holds for all sequences $(\omega_n)_{n\ge1}$ in a subset of $\prod_{n=1}^{\infty}\Omega_n$ that occurs in probability. Thus, for instance, it suffices to assume that (*) holds for all sequences in $\prod_{n=1}^{\infty}A_n$, where $A_n\in\mathcal{A}_n$ is such that $P_n(A_n)=1$, for all $n\ge1$.

Remark 19. One can easily obtain a version of Proposition 17 whose thesis states that a certain sequence $(U_n)_{n\ge1}$ of random objects is tight. To this aim, delete the sequences $(X''_n)_{n\ge1}$, $(Y''_n)_{n\ge1}$, $(\mathcal{X}''_n)_{n\ge1}$ and $(\mathcal{Y}''_n)_{n\ge1}$ from the statement of Proposition 17, add a topological space $\mathcal{U}$ that is homeomorphic to a Borel subset of a Polish space and a sequence of $\mathcal{U}$-valued random objects $(U_n)_{n\ge1}$. Assume that all $(\Omega_n,\mathcal{A}_n)$ are standard Borel and replace condition (*) with:

- if $X_{\lambda,n}(\omega_n) = O\big(Y_{\lambda,n}(\omega_n)\big)$ for all $\lambda\in\Lambda$, $X'_{\lambda,n}(\omega_n) = o\big(Y'_{\lambda,n}(\omega_n)\big)$ for all $\lambda\in\Lambda'$ and $\{Z_{\gamma,n}(\omega_n) : n\ge1\}$ is contained in a compact subset of $\mathcal{Z}_\gamma$ for all $\gamma\in\Gamma$ then $\{U_n(\omega_n) : n\ge1\}$ is contained in a compact subset of $\mathcal{U}$.

References

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