EXERCÍCIOS DE MATEMÁTICA DISCRETA
2o. SEMESTRE DE 2003

[Os exercícios desta lista são reciclados de edições anteriores desta e disciplinas semelhantes. Alguns exercícios estão em sua versão original, em inglês. (Melhor assim do que demorar mais ainda para eu disponibilizá-los).]

1. Let $\mathcal{A} = \{A_\lambda\colon \lambda \in \Lambda\}$ be a family of subsets of a finite set W. Consider the subsets $\Lambda' \subset \Lambda$ such that the family of sets A_λ ($\lambda \in \Lambda'$) admits a system of distinct representatives. Show that any two such *maximal* sets (under inclusion) have the same cardinality.
2. Let $M_i = M_i(S_i)$ ($1 \leq i \leq k$) be matroids on pairwise disjoint sets S_i. Let
$$\mathcal{I} = \Big\{ \bigcup_{1\leq i\leq k} I_i \colon I_i \in \mathcal{I}(M_i)\Big\} \tag{1}$$
be the family of unions of independent sets in the matroids M_i (one independent set I_i for each i). Clearly, $\mathcal{I} \subset 2^S$, where $S = \bigcup_{1\leq i\leq k} S_i$.
   (i) Show that $\mathcal{I}$ is the set of independent sets of a matroid on S. This matroid is called the *direct sum* of the matroids M_i ($1 \leq i \leq k$), and is usually denoted $\bigoplus_{1\leq i\leq k} M_i$.
   (ii) Determine the rank function of $\bigoplus_{1\leq i\leq k} M_i$.
3. Let $f\colon S \to T$ be a surjection, where S is a finite set. Suppose $M = M(S)$ is a matroid on S. Let
$$\mathcal{I} = \{f(I)\colon I \in \mathcal{I}(M)\} \tag{2}$$
be the family of images of independent sets in M under f. Clearly, $\mathcal{I} \subset 2^T$.
   (i) Show that $\mathcal{I}$ is the set of independent sets of a matroid on T. This matroid is sometimed called the *homomorphic image* of M under f.
   (ii) Let $N = N_f$ be the homomorphic image of M under f. Show that the rank function r_N of N is given by
$$r_N(A) = \min\{r_M(f^{-1}(B)) + |A \setminus B|\colon B \subset A\} \tag{3}$$
   for all $A \subset T$, where r_M is the rank function of M.
4. [Horn] Let k be a positive integer. Suppose $x_1, \dots, x_n$ are vectors in some vector space such that, for any $J \subset [n]$, there is $J' \subset J$ such that $|J'| \geq |J|/k$ and the vectors x_j ($j \in J'$) are linearly independent. Prove that $x_1, \dots, x_n$ may be partitioned into at most k families of linearly independent vectors.

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5. [Nash-Williams] Let G be a multigraph with no loops and at least one edge. Prove that the edge set of G may be written as a union of k forests if and only if

$$k \geq \frac{|E(H)|}{|V(H)| - c(H)} \tag{4}$$

for all subgraphs $H \subset G$ of G with $|V(H)| - c(H) > 0$, where $c(H)$ denotes the number of components in the graph H.

6. [Tutte] Let G be a connected graph. Prove that G contains k edge-disjoint spanning trees if and only if for any partition π of the vertex set of G, the number of edges joining vertices in distinct blocks of π is at least $k(|\pi| - 1)$, where $|\pi|$ denotes the number of blocks in π.

7. [Edmonds] Let $M = M(S)$ and $N = N(S)$ be two matroids on S. Prove that

$$\begin{aligned} \max\{|I| \colon I \in \mathcal{I}(M) \cap \mathcal{I}(N)\} \\ = \min\{r_M(A) + r_N(S \setminus A) \colon A \subset S\}. \end{aligned} \tag{5}$$

[*Sugestão.* Consider the matroid $M \vee N^*$, where N^* is the dual of N.]

8. Let G be a connected graph, and suppose that the edges of G are coloured in an arbitrary way. Prove that G contains a spanning tree with all its edges coloured with different colours if and only if, for all subgraphs $H \subset G$ of G, the number of colours that occur in $E(G) \setminus E(H)$ is at least $c(H) - 1$, where, again, $c(H)$ denotes the number of components in H. [*Sugestão.* Observe that, if S is a finite set, then the family of all sets of cardinality at most m is the family of independent sets of a matroid (this is the *uniform matroid* $U_{n,m}$, where $n = |S|$). Use this observation for $m = 1$ and S the colour classes of the edge colouring of G. Apply the above theorem of Edmonds. Alternatively, give a proof from first principles.]