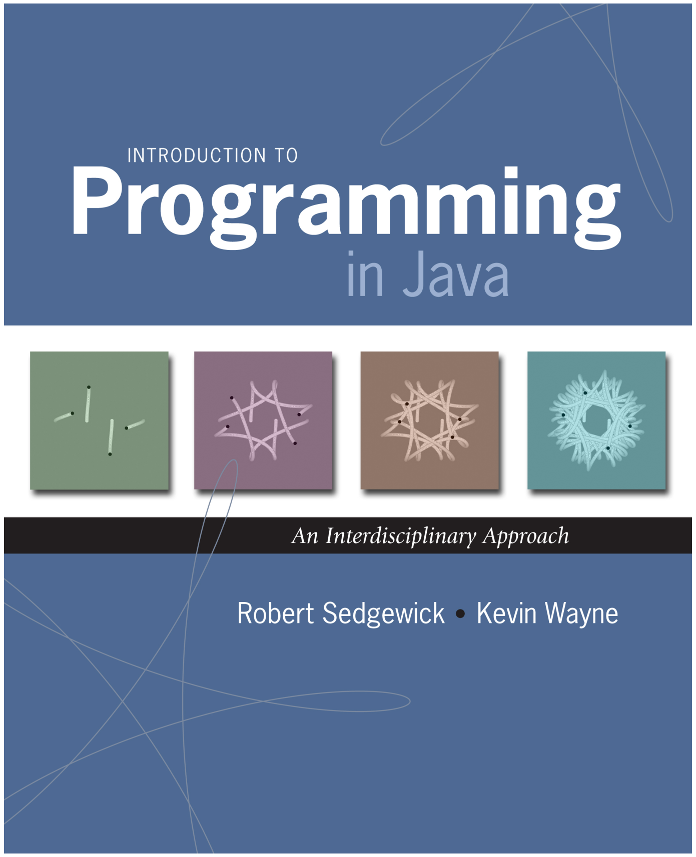
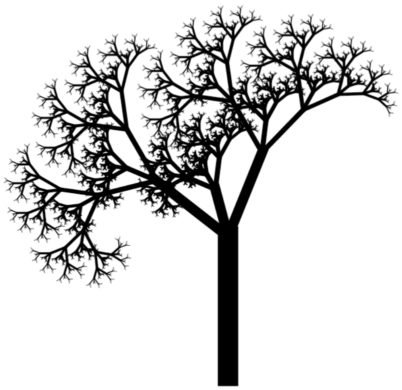
2.3 Recursion



Overview

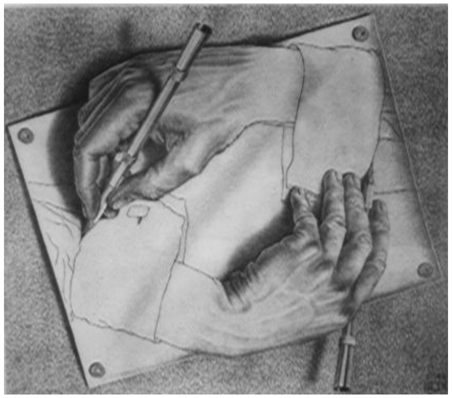
What is recursion? When one function calls itself directly or indirectly.



Why learn recursion?
- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.
- Mergesort, FFT, gcd, depth-first search.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.



Reproductive Parts
M. C. Escher, 1948

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$$4032 = 2^6 \times 3^2 \times 7^1$$
$$1272 = 2^3 \times 3^1 \times 53^1$$
$$\text{gcd} = 2^3 \times 3^1 = 24$$

Applications.

- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q, p \% q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

```
gcd(4032, 1272)   = gcd(1272, 216)
                  = gcd(216, 192)
                  = gcd(192, 24)
                  = gcd(24, 0)
                  = 24.
```

4032 = 3 × 1272 + 216

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \% q) & \text{otherwise} \end{cases}$$

← base case

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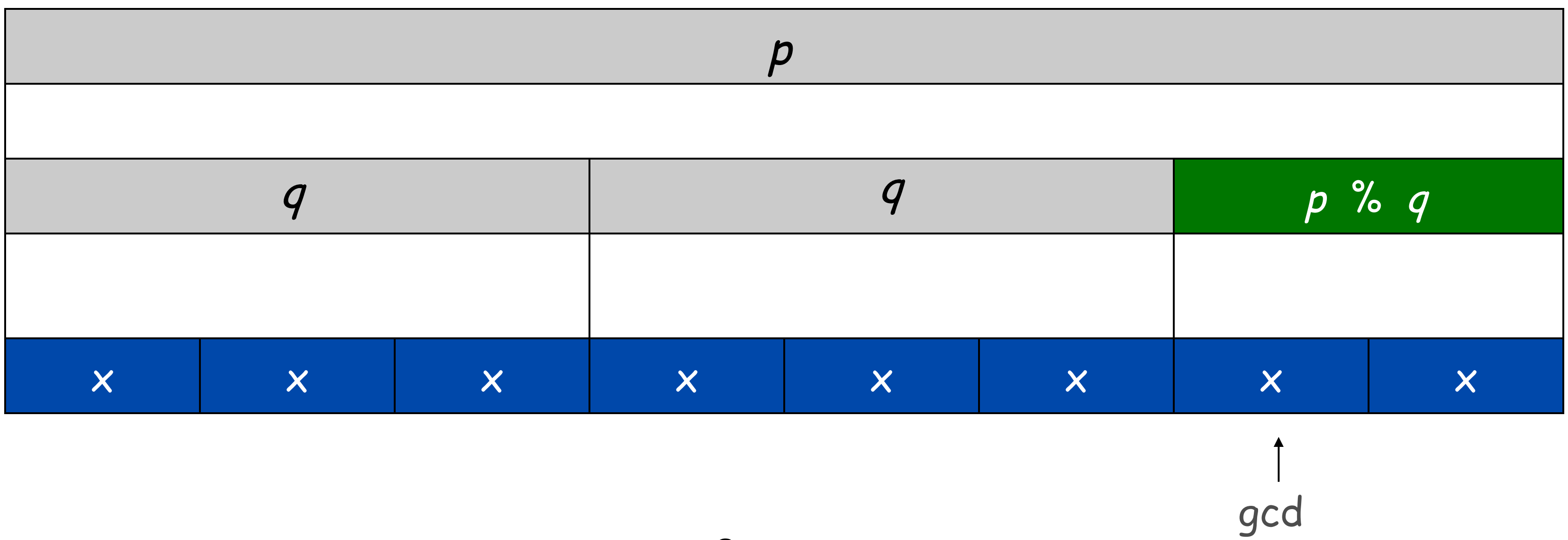


p = 8x
q = 3x
gcd(p, q) = x

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \,\%\, q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

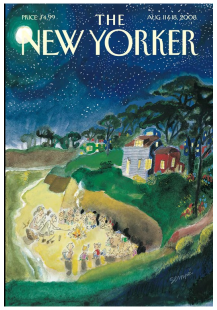
Java implementation.

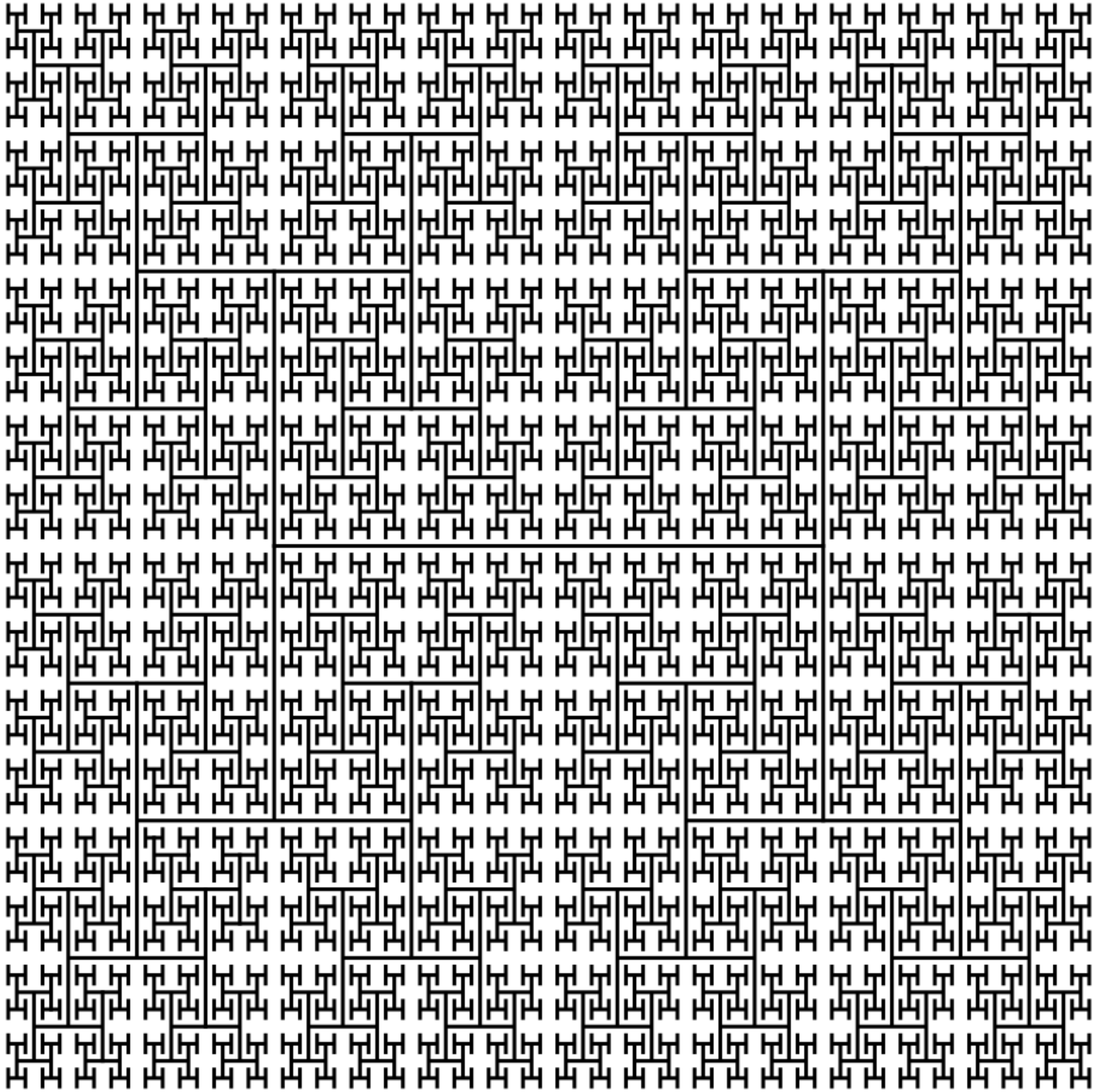
```
public static int gcd(int p, int q) {
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

← base case

← reduction step

Recursive Graphics



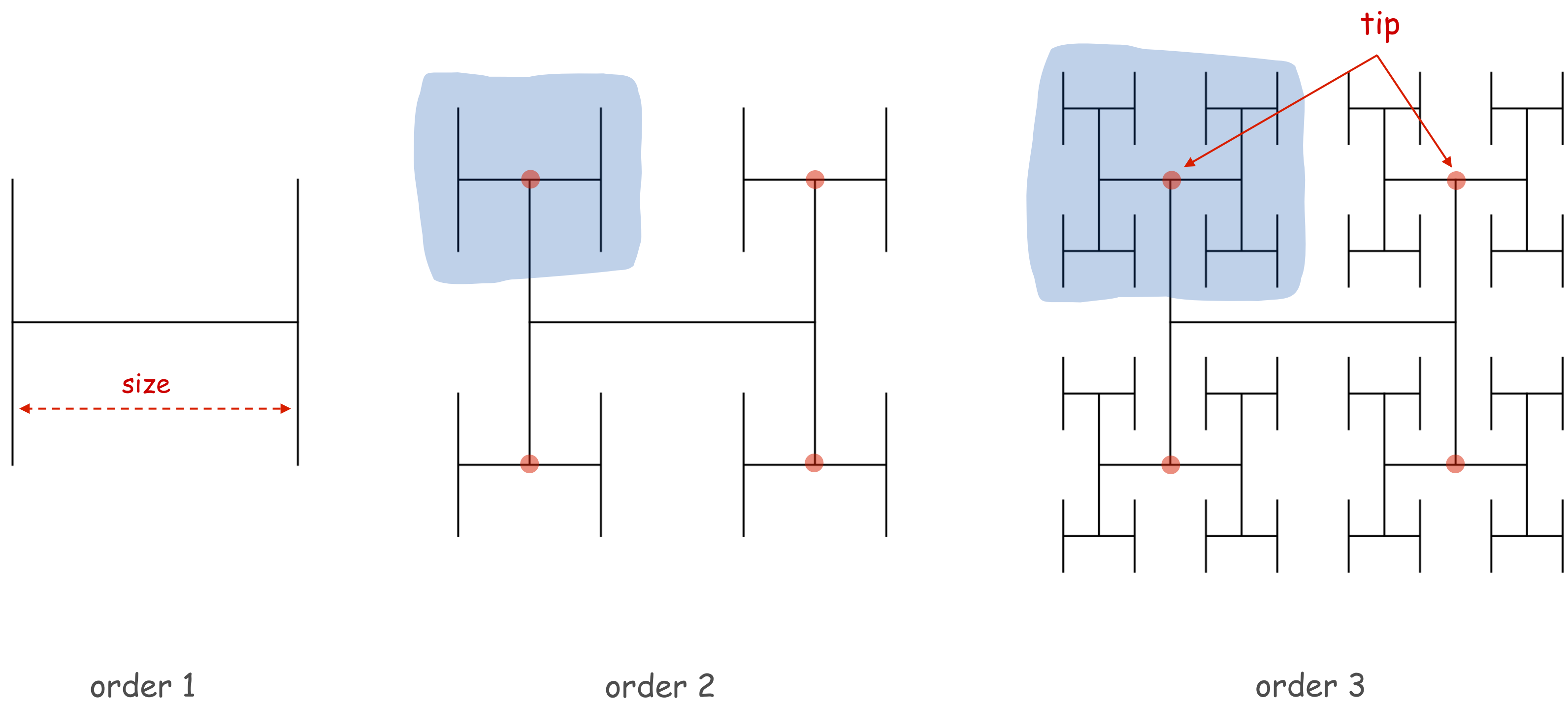


Htree

H-tree of order n.

- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip.

and half the size

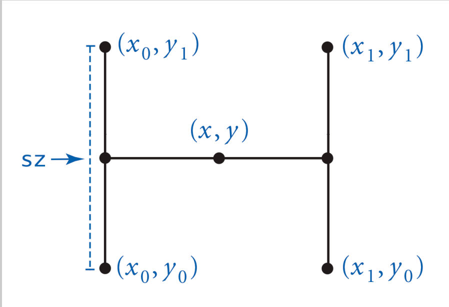


Htree in Java

```
public class Htree {
   public static void draw(int n, double sz, double x, double y) {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;

      StdDraw.line(x0,  y, x1,  y);      // draw the H, centered on (x, y)
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, sz/2, x0, y0);           // recursively draw 4 half-size Hs
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

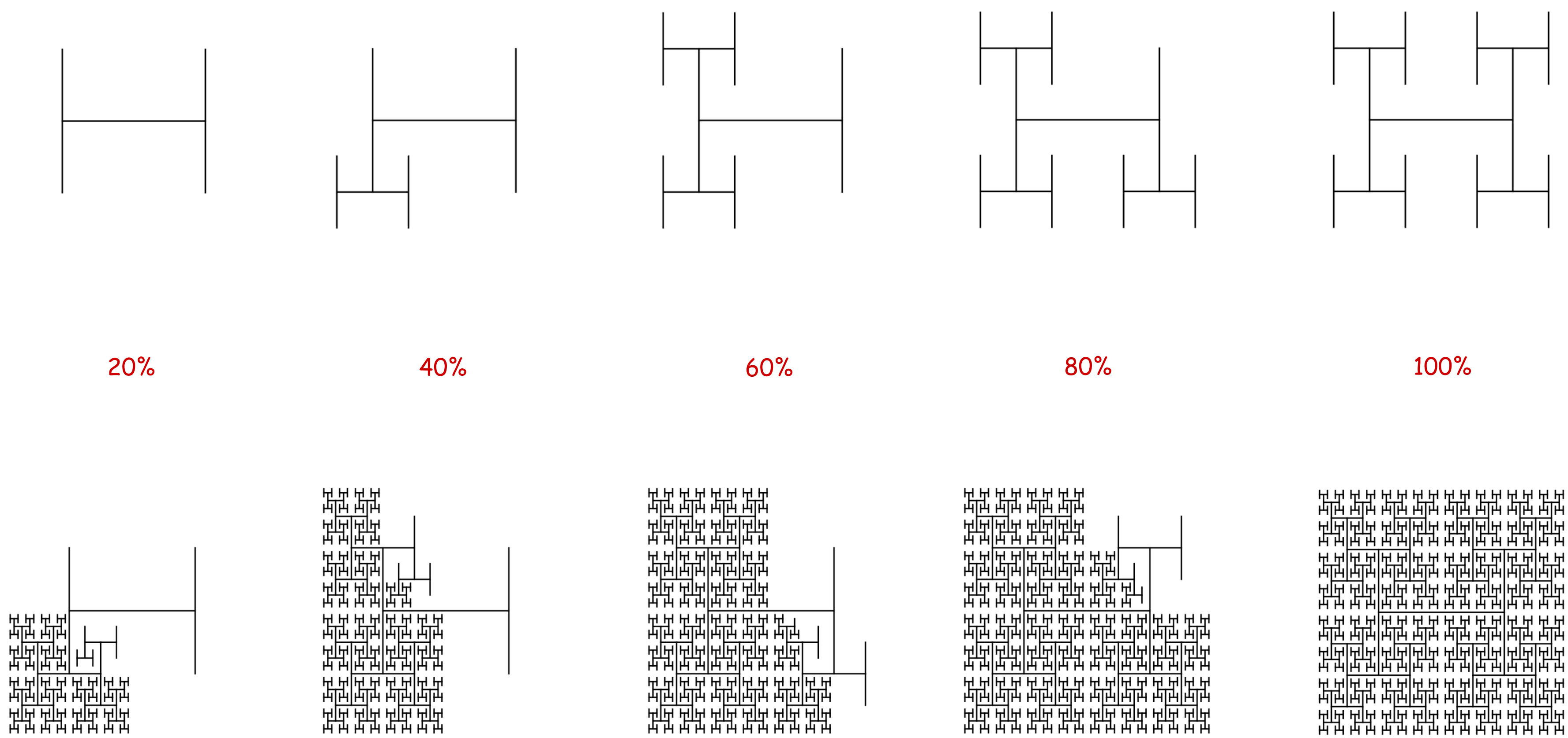
← draw the H, centered on (x, y)

← recursively draw 4 half-size Hs

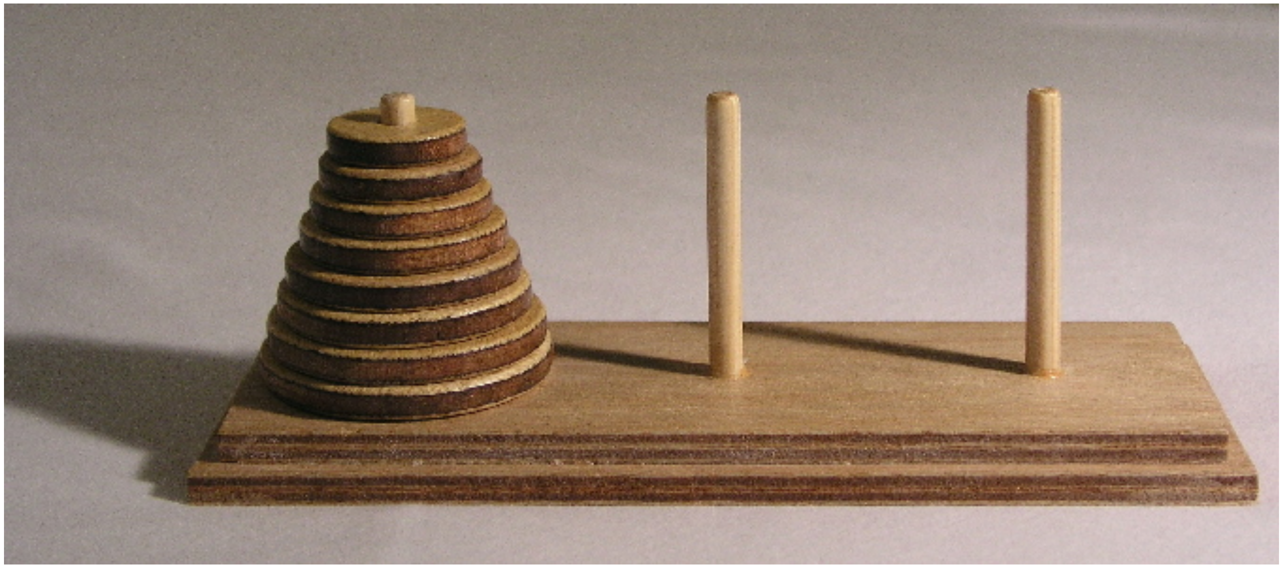


Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.



Towers of Hanoi

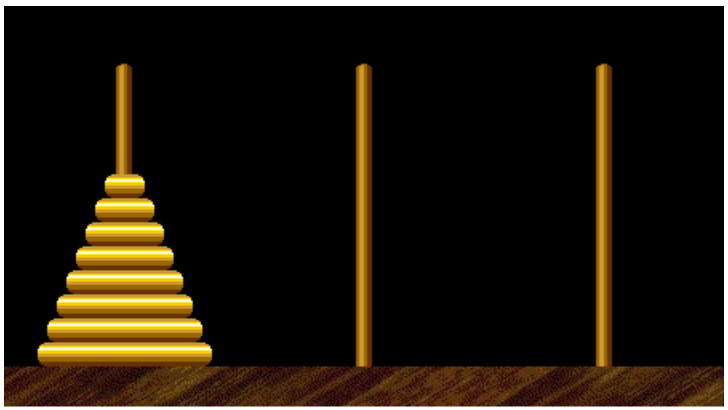


http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

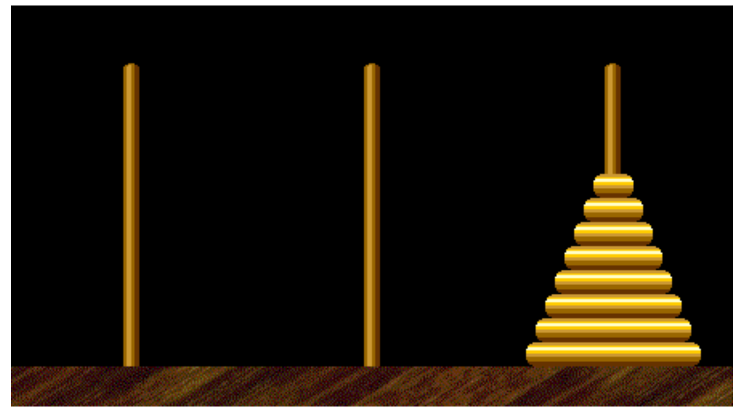
Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



start



finish

Towers of Hanoi demo

Edouard Lucas (1883)

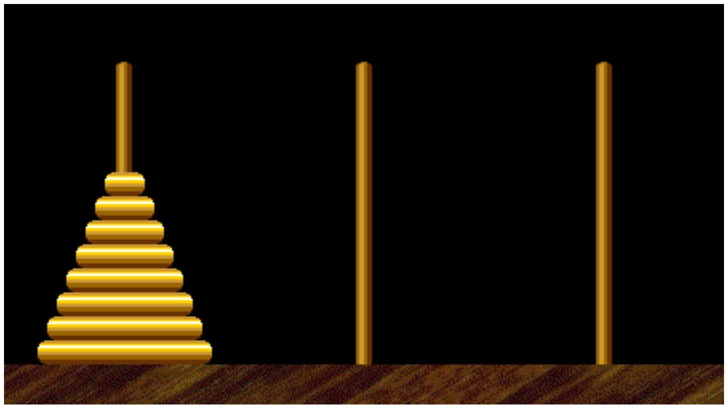
Towers of Hanoi Legend

Q. Is world going to end (according to legend)?

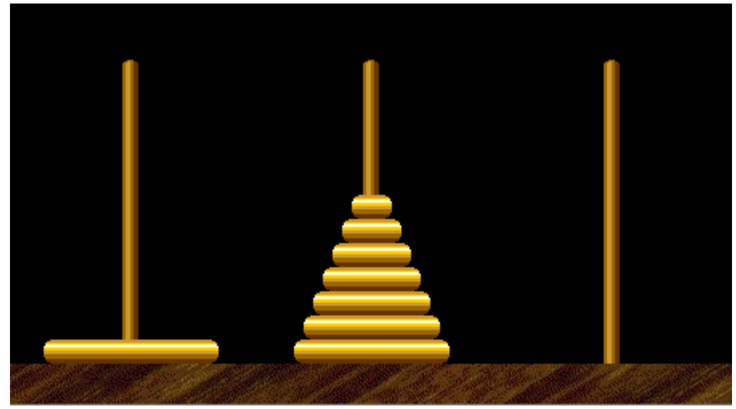
- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

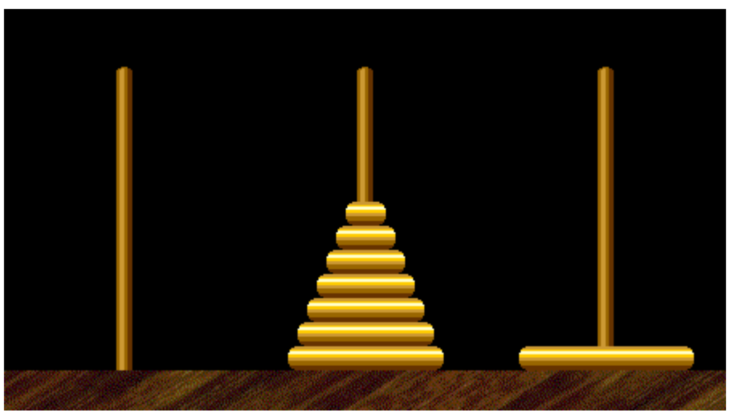


Move n-1 smallest discs right.

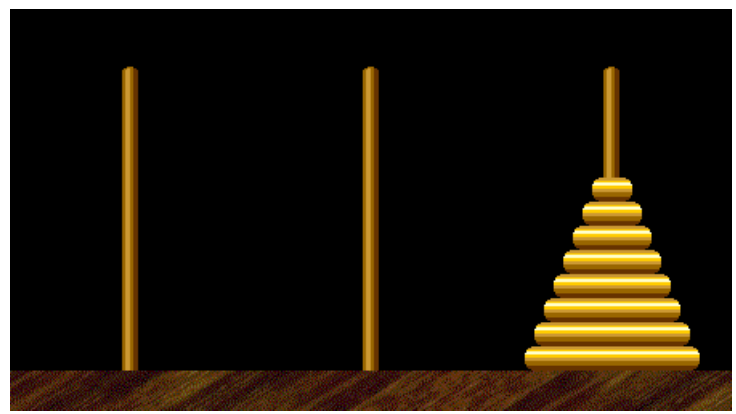


Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.



Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi {

   public static void moves(int n, boolean left) {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

`moves(n, true)` : move discs 1 to n one pole to the left
`moves(n, false)`: move discs 1 to n one pole to the right

smallest disc

Towers of Hanoi: Recursive Solution

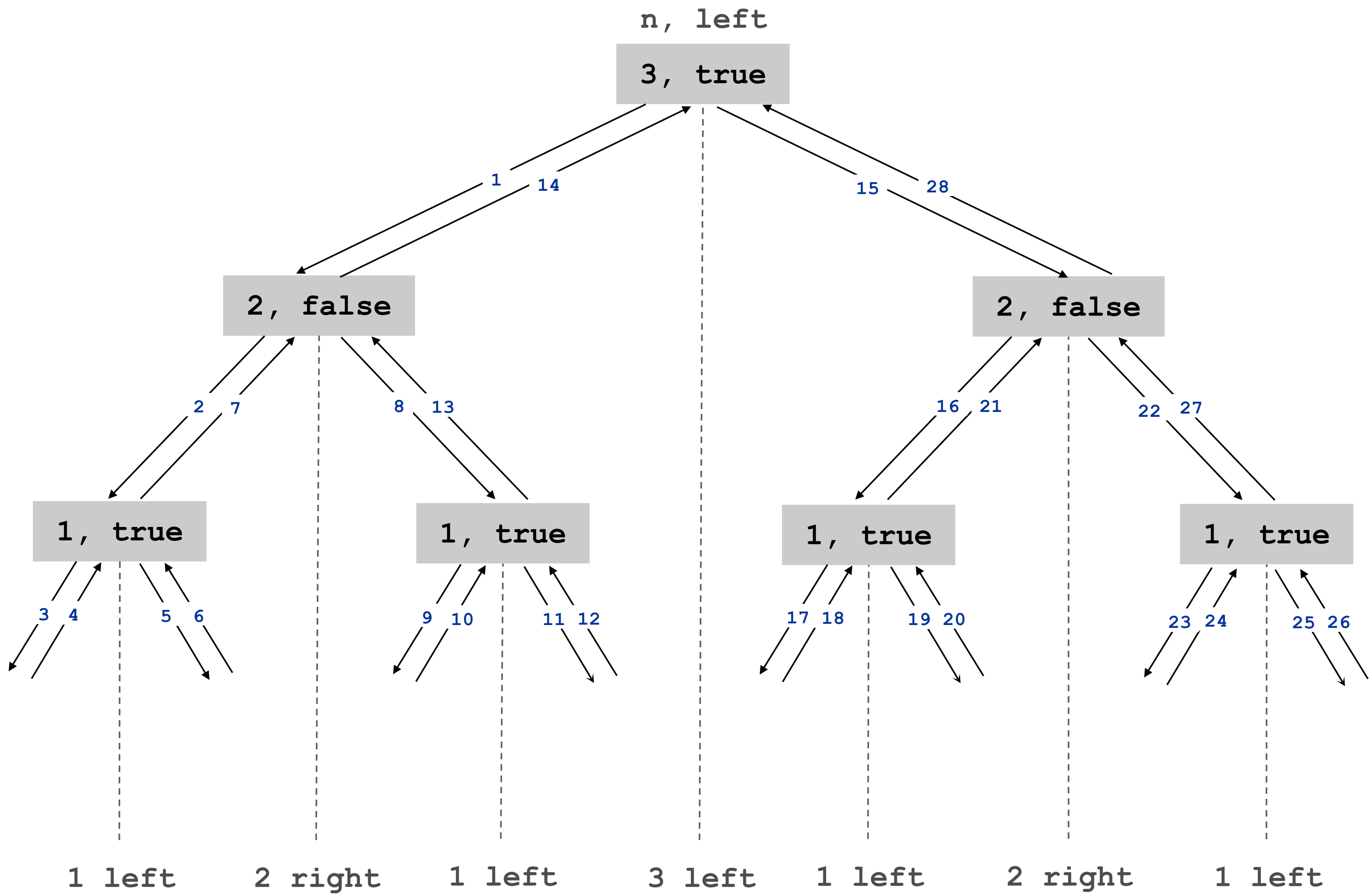
```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions of ruler

Towers of Hanoi: Recursion Tree



Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.

- Takes $2^n - 1$ moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!

- Alternate between two moves:
  - move smallest disc to right if n is even (to left if n is odd)
  - make only legal move not involving smallest disc

Recursive algorithm may reveal fate of world.

- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Divide-and-Conquer

Divide-and-conquer paradigm.

- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

> Divide et impera. Veni, vidi, vici. - Julius Caesar

Many important problems succumb to divide-and-conquer.

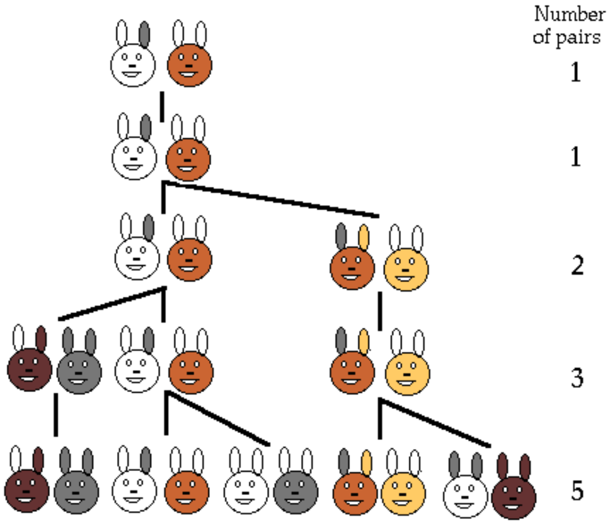
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.
- Midpoint displacement method for fractional Brownian motion.

Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$



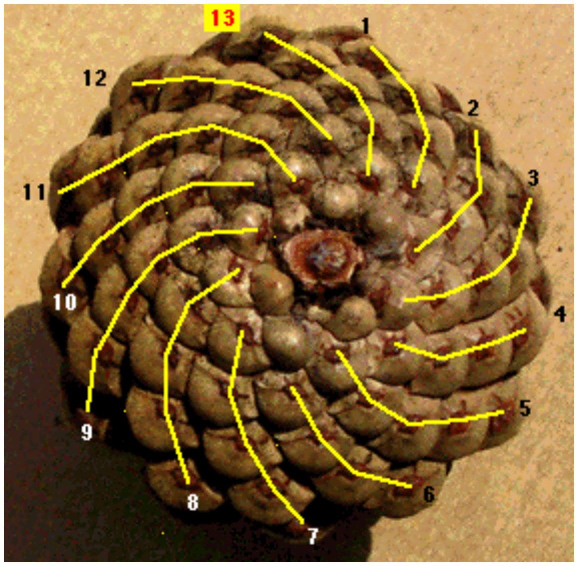
Fibonacci rabbits

L. P. Fibonacci
(1170 - 1250)

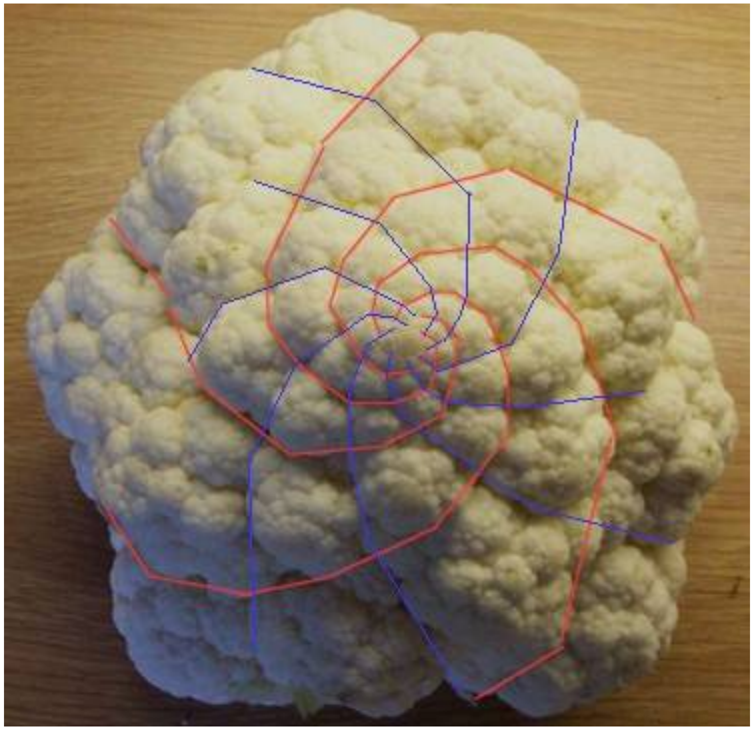
Fibonacci Numbers and Nature

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$



pinecone



cauliflower

A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$

A natural for recursion?

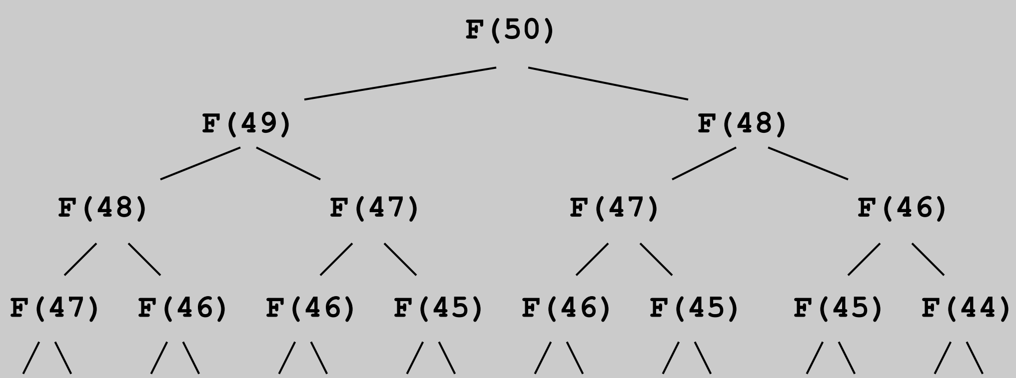
```
public static long F(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

Recursion Challenge 1 (difficult but important)

Q. Is this an efficient way to compute F(50)?

```
public static long F(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

A. No, no, no! This code is spectacularly inefficient.



recursion tree for naïve Fibonacci function

F(50) is called once.

F(49) is called once.

F(48) is called 2 times.

F(47) is called 3 times.

F(46) is called 5 times.

F(45) is called 8 times.

...

F(1) is called 12,586,269,025 times.

F(50)

Recursion Challenge 2 (easy and also important)

Q. Is this a more efficient way to compute F(50)?

```
public static long F(int n) {
   if (n == 0) return 0;
   long[] F = new long[n+1];
   F[0] = 0;
   F[1] = 1;
   for (int i = 2; i <= n; i++)
      F[i] = F[i-1] + F[i-2];
   return F[n];
}
```

FYI: classic math

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rceil$$

ϕ = golden ratio $\approx$ 1.618

A. Yes. This code does it with 50 additions.

Lesson. Don't use recursion to engage in exponential waste.

Context. This is a special case of an important programming technique known as dynamic programming (stay tuned).

Summary

How to write simple recursive programs?

- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Towers of Hanoi by W. A. Schloss.

Why learn recursion?

- New mode of thinking.
- Powerful programming tool.

Divide-and-conquer. Elegant solution to many important problems.