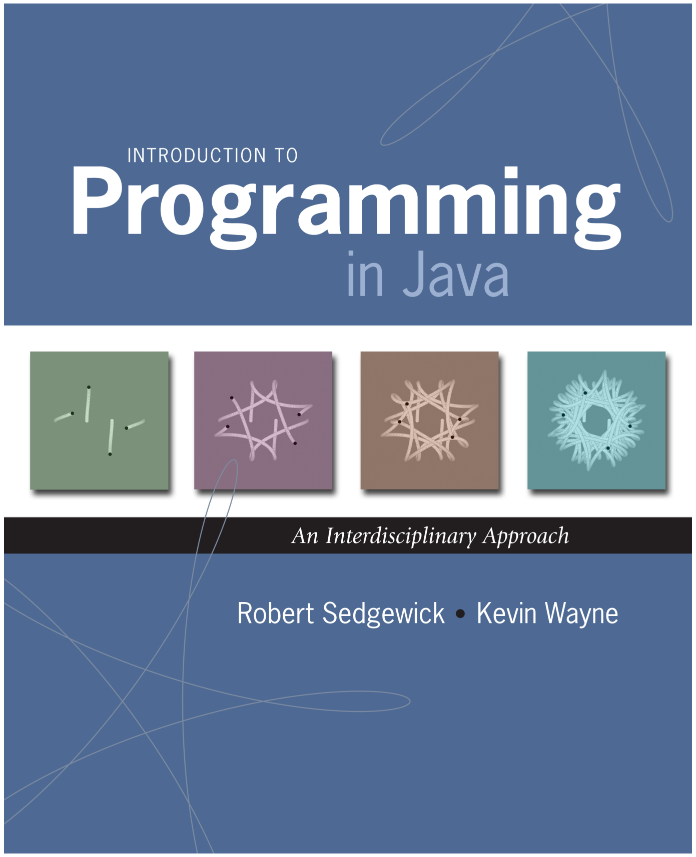
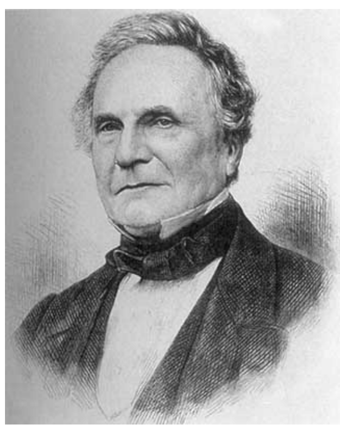
4.1 Performance

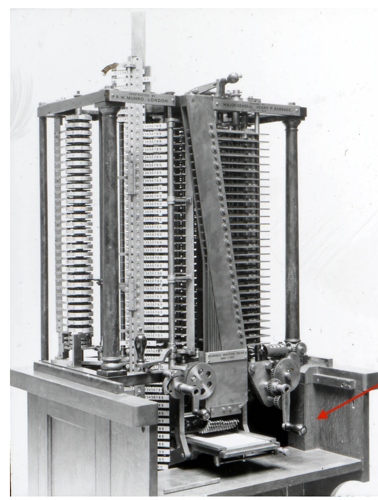


Running Time

> *"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"* – *Charles Babbage*



Charles Babbage (1864)



Analytic Engine

how many times do you have to turn the crank?

The Challenge

Q. Will my program be able to solve a large practical problem?

compile

debug on test case

solve problems in practice

Key insight. [Knuth 1970s]
Use the scientific method to understand performance.

Scientific Method

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.

- Experiments we design must be reproducible.
- Hypothesis must be falsifiable.

Reasons to Analyze Algorithms

Predict performance.

- Will my program finish?
- When will my program finish?

Compare algorithms.

- Will this change make my program faster?
- How can I make my program faster?

Basis for inventing new ways to solve problems.

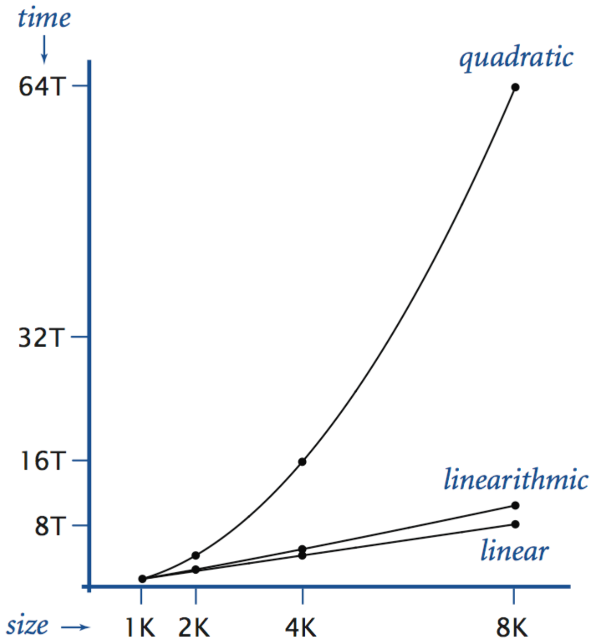
- Enables new technology.
- Enables new research.

Algorithmic Successes

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ...
- Brute force: N^2 steps.
- FFT algorithm: N log N steps, enables new technology.

Freidrich Gauss
1805

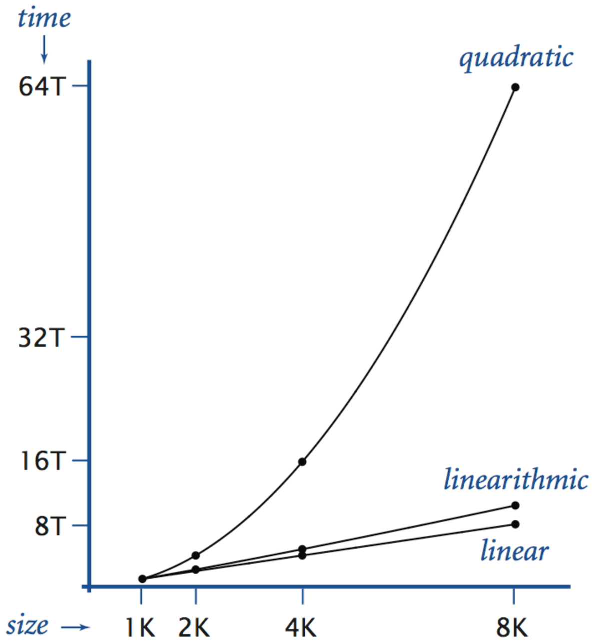


Algorithmic Successes

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: N log N steps, enables new research.

Andrew Appel
PU '81



Three-Sum Problem

Three-sum problem. Given N integers, find triples that sum to 0.
Context. Deeply related to problems in computational geometry.

```
% more 8ints.txt
30 -30 -20 -10 40 0 10 5

% java ThreeSum < 8ints.txt
 4
 30 -30   0
 30 -20 -10
-30 -10  40
-10   0  10
```

Q. How would you write a program to solve the problem?

Three-Sum: Brute-Force Solution

```java
public class ThreeSum {

   // return number of distinct triples (i, j, k)
   // such that (a[i] + a[j] + a[k] == 0)
   public static int count(int[] a) {
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0) cnt++;
      return cnt;
   }

   public static void main(String[] args) {
      int[] a = StdArrayIO.readInt1D();
      StdOut.println(count(a));
   }
}
```

all possible triples i < j < k

Empirical Analysis

Empirical Analysis

Empirical analysis. Run the program for various input sizes.

N	*time* †
512	0.03
1024	0.26
2048	2.16
4096	17.18
8192	136.76

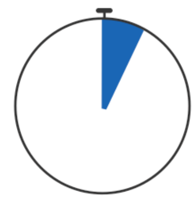
† Running Linux on Sun-Fire-X4100 with 16GB RAM

Stopwatch

Q. How to time a program?

A. A stopwatch.

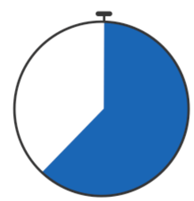
```
% java ThreeSum < 1Kints.txt
```



tick tick tick

```
0
```

```
% java ThreeSum < 2Kints.txt
```



tick tick tick tick tick tick
tick tick tick tick tick tick
tick tick tick tick tick tick
tick tick tick tick tick tick

```
2
391930676 -763182495 371251819
-326747290 802431422 -475684132
```

Stopwatch

Q. How to time a program?

A. A `Stopwatch` object.

public class Stopwatch		
	Stopwatch()	*create a new stopwatch and start it running*
double	elapsedTime()	*return the elapsed time since creation, in seconds*

```
public class Stopwatch {
   private final long start;

   public Stopwatch() {
      start = System.currentTimeMillis();
   }

   public double elapsedTime() {
      return (System.currentTimeMillis() - start) / 1000.0;
   }
}
```

Stopwatch

Q. How to time a program?

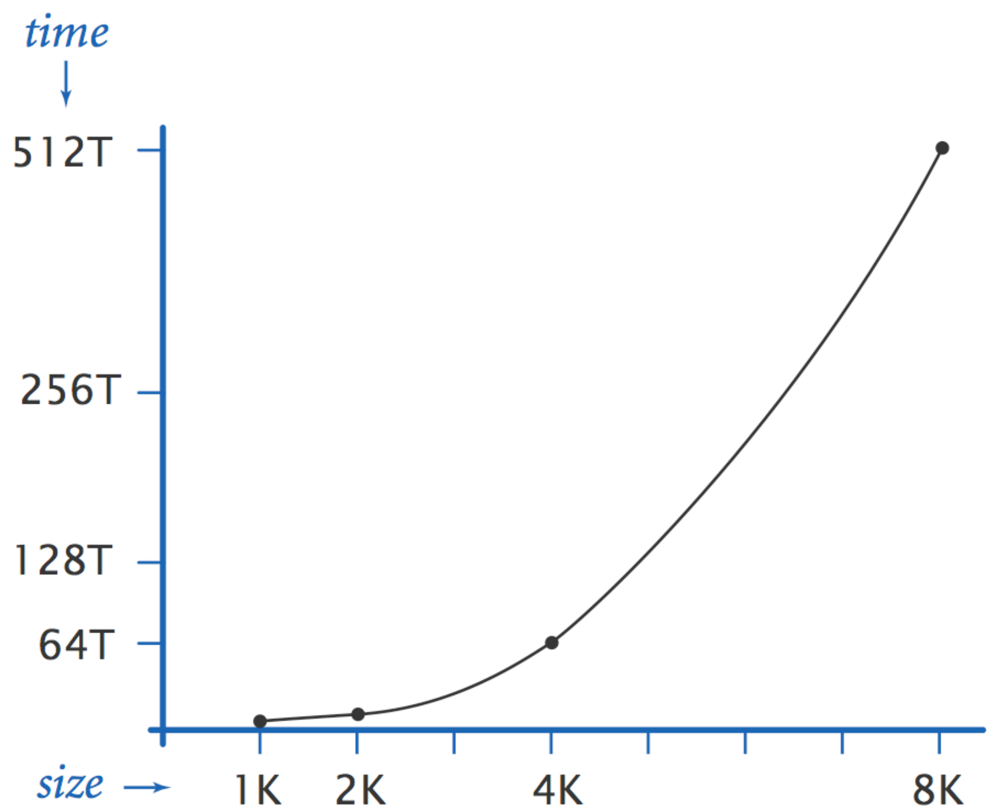
A. A **Stopwatch** object.

public class Stopwatch		
	Stopwatch()	*create a new stopwatch and start it running*
double	elapsedTime()	*return the elapsed time since creation, in seconds*

```
public static void main(String[] args) {
   int[] a = StdArrayIO.readInt1D();
   Stopwatch timer = new Stopwatch();
   StdOut.println(count(a));
   StdOut.println(timer.elapsedTime());
}
```

Empirical Analysis

Data analysis. Plot running time vs. input size N.



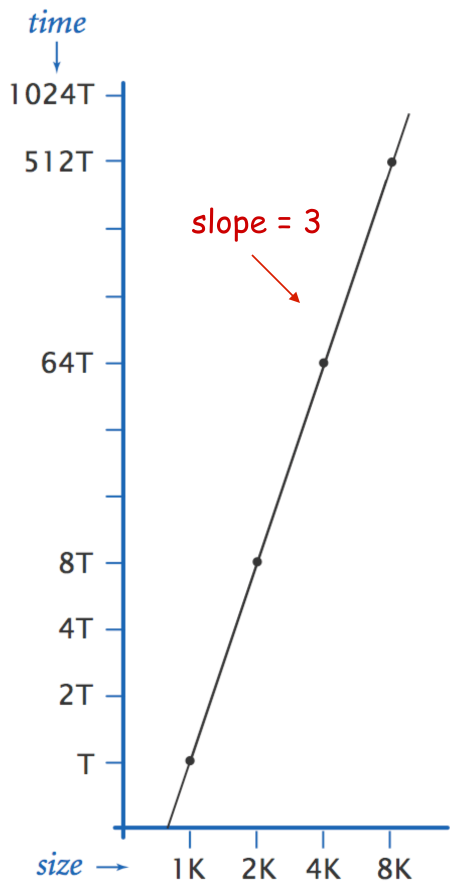
Q. How fast does running time grow as a function of input size N?

Empirical Analysis

Initial hypothesis. Running time obeys power law $f(N) = a\,N^{\,b}$.

Data analysis. Plot running time vs. input size N on a log-log scale.

Consequence. Power law yields straight line (slope = b).



slope

Refined hypothesis. Running time grows as cube of input size: $a\,N^{3}$.

Doubling Hypothesis

Doubling hypothesis. Quick way to estimate b in a power law hypothesis.

Run program, doubling the size of the input?

N	time †	ratio
512	0.033	-
1024	0.26	7.88
2048	2.16	8.43
4096	17.18	7.96
8192	136.76	7.96

seems to converge to a constant $c = 8$

Hypothesis. Running time is about $a\,N^b$ with $b = lg\ c$.

Performance Challenge 1

Let F(N) be running time of **`main()`** as a function of input N.

```
public static void main(String[] args) {
   ...
   int N = Integer.parseInt(args[0]);
   ...
}
```

Scenario 1. F(2N) / F(N) converges to about 4.

Q. What is order of growth of the running time?

Performance Challenge 2

Let F(N) be running time of **`main()`** as a function of input N.

```
public static void main(String[] args) {
    ...
    int N = Integer.parseInt(args[0]);
    ...
}
```

Scenario 2. F(2N) / F(N) converges to about 2.

Q. What is order of growth of the running time?

Prediction and Validation

Hypothesis. Running time is about $a\,N^3$ for input of size N.

Q. How to estimate a?

A. Run the program!

N	*time* †
4096	17.18
4096	17.15
4096	17.17

$17.17 = a\ 4096^3$

$\Rightarrow\ a = 2.5 \times 10^{-10}$

Refined hypothesis. Running time is about $2.5 \times 10^{-10} \times N^3$ seconds.

Prediction. 1,100 seconds for $N = 16{,}384$.

Observation.

N	*time* †
16384	1118.86

← *validates hypothesis!*

Mathematical Analysis

Donald Knuth
Turing award '74

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0) count++;
```

operation	frequency
variable declaration	2
variable assignment	2
less than comparison	$N + 1$
equal to comparison	N
array access	N
increment	$\leq 2N$

between N (no zeros)
and 2N (all zeros)

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         if (a[i] + a[j] == 0) count++;
```

operation	frequency
variable declaration	$N + 2$
variable assignment	$N + 2$
less than comparison	$1/2\ (N+1)\ (N+2)$
equal to comparison	$1/2\ N\ (N-1)$
array access	$N\ (N-1)$
increment	$\leq\ N^2$

$$0\ +\ 1\ +\ 2\ +\ \ldots\ +\ (N-1)\ =\ 1/2\ \ N(N-1)$$

becoming very tedious to count

Tilde Notation

Tilde notation.

- Estimate running time as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

Ex 1. $6N^3 + 17N^2 + 56 \sim 6N^3$

Ex 2. $6N^3 + 100N^{4/3} + 56 \sim 6N^3$

Ex 3. $6N^3 + \underbrace{17N^2 \log N}_{\text{discard lower-order terms}} \sim 6N^3$

discard lower-order terms
(e.g., N = 1000: 6 trillion vs. 169 million)

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
public static int count(int[] a)
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
               cnt++;
   return cnt;
}
```

inner loop

1

N

$\sim N^2/2$

$\sim N^3/6$

depends on input data

Inner loop. Focus on instructions in "inner loop."

Constants in Power Law

Power law. Running time of a typical program is $\sim a\, N^b$.

Exponent b depends on: algorithm.

Leading constant a depends on:

- Algorithm.
- Input data.

system independent effects

- Caching.
- Machine.
- Compiler.
- Garbage collection.
- Just-in-time compilation.
- CPU use by other applications.

system dependent effects

Our approach. Use doubling hypothesis (or mathematical analysis) to estimate exponent b, run experiments to estimate a.

Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

Order of Growth Classifications

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

$\lg N$

$\lg N = \log_2 N$

```
for (int i = 0; i < N; i++)
   ...
```

N

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

N^2

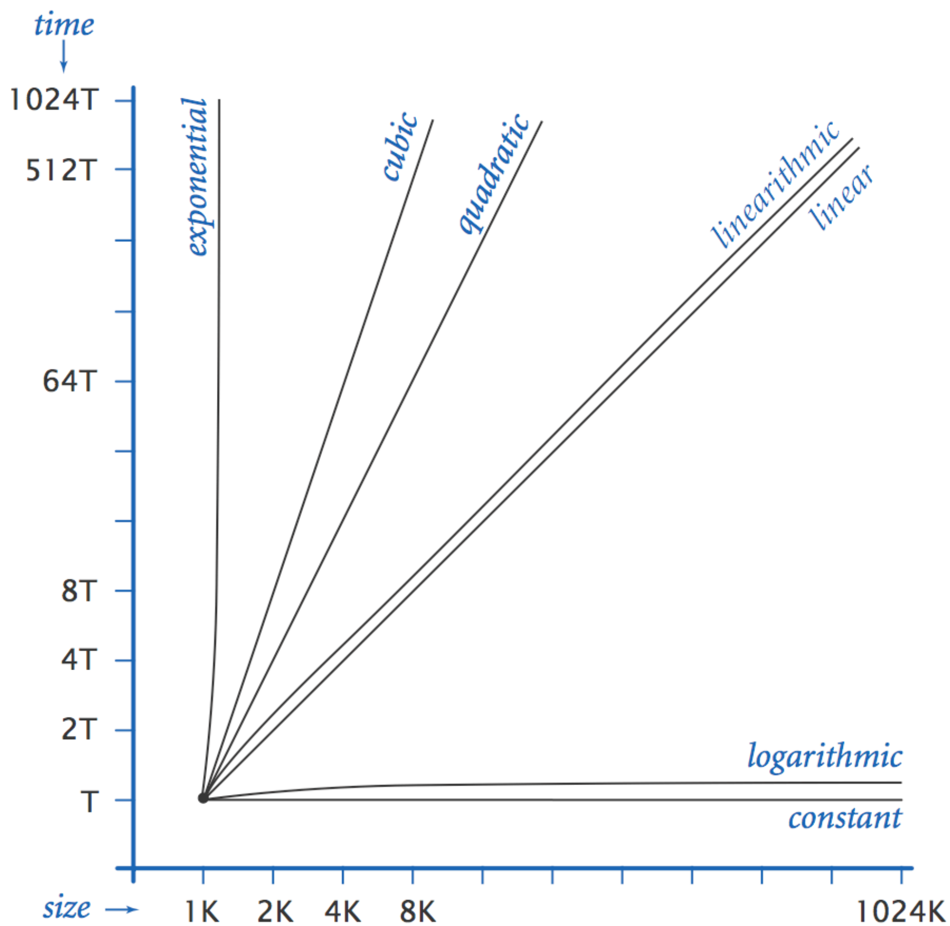
```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

$N \lg N$

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

2^N

Order of Growth Classifications



Orders of growth (log-log plot)

order of growth description	*order of growth* function	*factor for doubling hypothesis*
constant	1	1
logarithmic	$\log N$	1
linear	N	2
linearithmic	$N \log N$	2
quadratic	N^2	4
cubic	N^3	8
exponential	2^N	2^N

Commonly encountered growth functions

Order of Growth: Consequences

order of growth	predicted running time if problem size is increased by a factor of 100
linear	a few minutes
linearithmic	a few minutes
quadratic	several hours
cubic	a few weeks
exponential	forever

Effect of increasing problem size for a program that runs for a few seconds

order of growth	predicted factor of problem size increase if computer speed is increased by a factor of 10
linear	10
linearithmic	10
quadratic	3-4
cubic	2-3
exponential	1

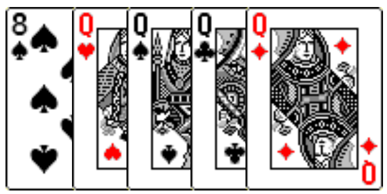
Effect of increasing computer speed on problem size that can be solved in a fixed amount of time

Dynamic Programming

Binomial Coefficients

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Ex. Number of possible 7-card poker hands = $\binom{52}{7}$ = 2,598,960.



Ex. Probability of "quads" in Texas hold 'em:

$$\frac{\binom{13}{1}\binom{4}{4} \times \binom{48}{3}}{\binom{52}{7}} = \frac{224{,}848}{133{,}784{,}560} \quad (about\ \ 594:1)$$

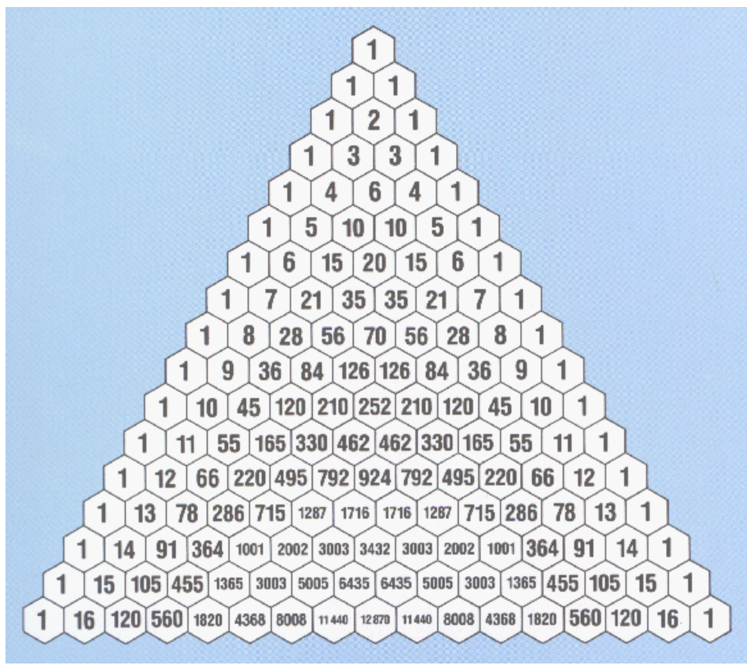
Binomial Coefficients

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Pascal's identity.

$$\binom{n}{k} = \underbrace{\binom{n-1}{k-1}}_{\text{contains first element}} + \underbrace{\binom{n-1}{k}}_{\text{excludes first element}}$$

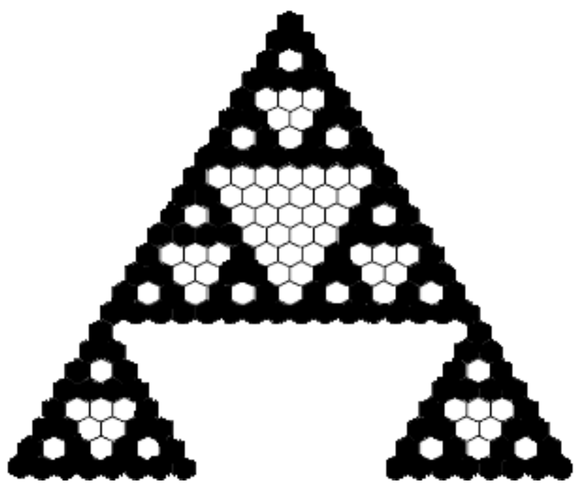
Pascal's triangle.



Binomial Coefficients: Sierpinski Triangle

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Sierpinski triangle. Color black the odd integers in Pascal's triangle.



Binomial Coefficients: First Attempt

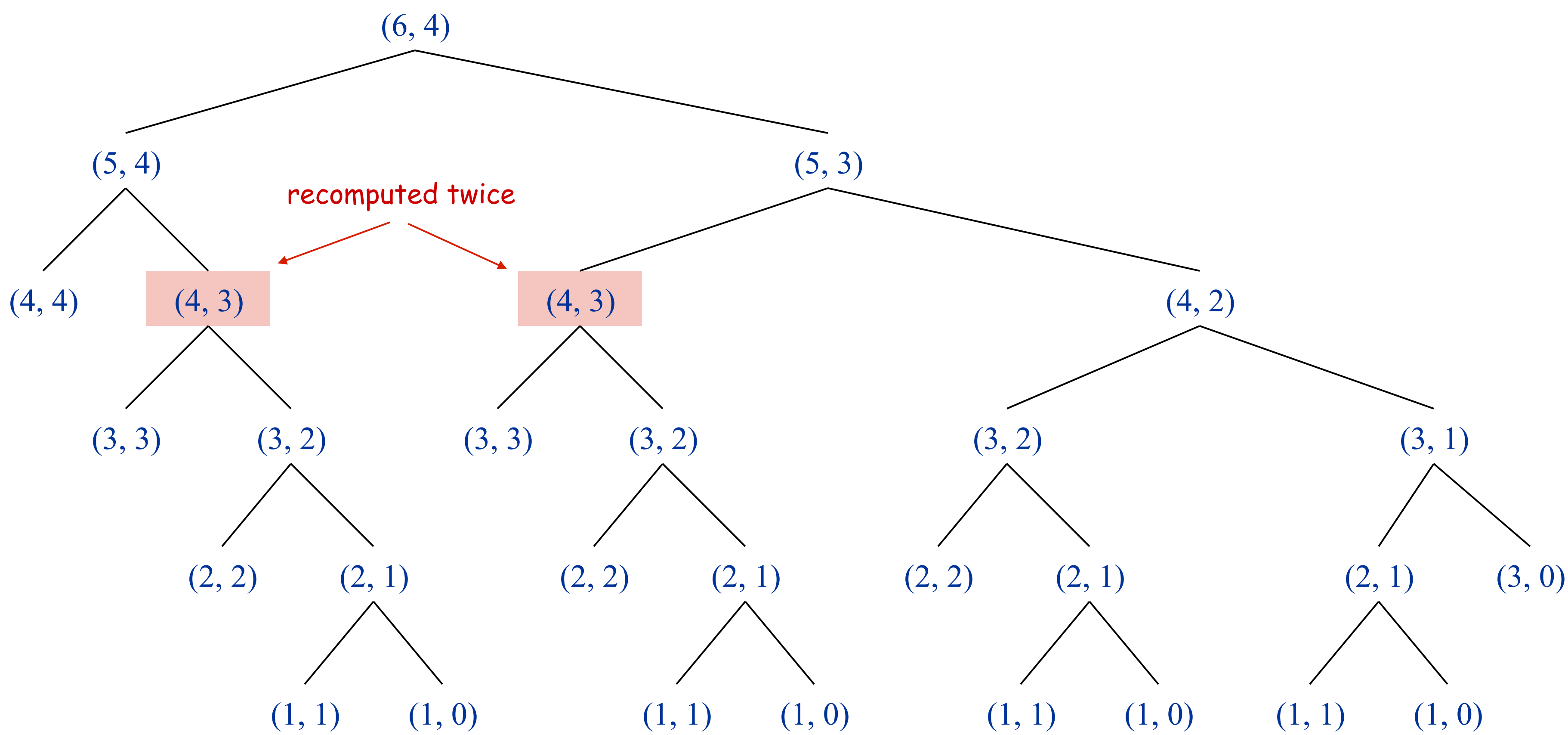
```java
public class SlowBinomial {

   // natural recursive implementation
   public static long binomial(long n, long k) {
      if (k == 0) return 1;
      if (n == 0) return 0;
      return binomial(n-1, k-1) + binomial(n-1, k);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      StdOut.println(binomial(N, K));
   }

}
```

Pascal's identity

Performance Challenge 3

Q. Is this an efficient way to compute binomial coefficients?

A. No, no, no! [same essential recomputation flaw as naïve Fibonacci]



Timing Experiments

Timing experiments: direct recursive solution.

$(2N, N)$	*time* †
(26, 13)	0.46
(28, 14)	1.27
(30, 15)	4.30
(32, 16)	15.69
(34, 17)	57.40
(36, 18)	230.42

increase N by 1, running time increases by about 4x

Q. Is running time linear, quadratic, cubic, exponential in N?

Performance Challenge 4

Let F(N) be running time to compute `binomial(2N, N)`.

```
public static long binomial(long n, long k) {
   if (k == 0) return 1;
   if (n == 0) return 0;
   return binomial(n-1, k-1) + binomial(n-1, k);
}
```

Observation. F(N+1) / F(N) converges to about 4.

Q. What is order of growth of the running time?

A. Exponential: a 4^N. ← will not finish unless N is small

Dynamic Programming

Key idea. Save solutions to subproblems to avoid recomputation.

n \ k	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	2	1	0	0
3	1	3	3	1	0
4	1	4	6	4	1
5	1	5	10	10	5
6	1	6	15	20	15

binomial(n, k)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$20 = 10 + 10$

Tradeoff. Trade (a little) memory for (a huge amount of) time.

Binomial Coefficients: Dynamic Programming

```
public class Binomial {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      long[][] bin = new long[N+1][K+1];

      // base cases
      for (int k = 1; k <= K; k++) bin[0][K] = 0;
      for (int n = 0; n <= N; n++) bin[N][0] = 1;

      // bottom-up dynamic programming
      for (int n = 1; n <= N; n++)
         for (int k = 1; k <= K; k++)
            bin[n][k] = bin[n-1][k-1] + bin[n-1][k];

      // print results
      StdOut.println(bin[N][K]);
   }
}
```

Timing Experiments

Timing experiments for binomial coefficients via dynamic programming.

$(2N, N)$	*time* †
(26, 13)	instant
(28, 14)	instant
(30, 15)	instant
(32, 16)	instant
(34, 17)	instant
(36, 18)	instant

Q. Is running time linear, quadratic, cubic, exponential in N?

Performance Challenge 5

Let F(N) be running time to compute `binomial(2N, N)` using DP.

```
for (int n = 1; n <= N; n++)
   for (int k = 1; k <= K; k++)
      bin[n][k] = bin[n-1][k-1] + bin[n-1][k];
```

Q. What is order of growth of the running time?

A. Quadratic: $a N^2$. ← effectively instantaneous for small N

Remark. There is a profound difference between 4^N and N^2.

cannot solve a large problem

can solve a large problem

Digression: Stirling's Approximation

Alternative: $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Caveat. 52! overflows a long, even though final result doesn't.

Instead of computing exact values, use Stirling's approximation:

$$\ln n! \approx n \ln n - n + \frac{\ln(2\pi n)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}$$

Application. Probability of exact k heads in n flips with a biased coin.

$$\binom{n}{k} p^k (1-p)^{n-k}$$ (easy to compute approximate value with Stirling's formula)

Memory

Typical Memory Requirements for Java Data Types

Bit. 0 or 1.

Byte. 8 bits.

Megabyte (MB). 1 million bytes ~ 2^{10} bytes.

Gigabyte (GB). 1 billion bytes ~ 2^{20} bytes.

type	*bytes*
`boolean`	1
`byte`	1
`char`	2
`int`	4
`float`	4
`long`	8
`double`	8

type	*bytes*
`int[]`	$4N + 16$
`double[]`	$8N + 16$
`Charge[]`	$36N + 16$
`int[][]`	$4N^2 + 20N + 16$
`double[][]`	$8N^2 + 20N + 16$
`String`	$2N + 40$

typical computer '10 has about 2GB memory

Q. What's the biggest **`double[]`** array you can store on your computer?

Performance Challenge 6

Q. How much memory does this program require as a function of N?

```
public class RandomWalk {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int[][] count = new int[N][N];
      int x = N/2;
      int y = N/2;

      for (int i = 0; i < N; i++)  {
         // no new variable declared in loop
         ...
         count[x][y]++;
      }
   }
}
```

A.

Summary

Q. How can I evaluate the performance of my program?
A. Computational experiments, mathematical analysis, scientific method.

Q. What if it's not fast enough? Not enough memory?

- Understand why.
- Buy a faster computer.
- Learn a better algorithm (COS 226, COS 423).
- Discover a new algorithm.

attribute	better machine	better algorithm
cost	$$$ or more	$ or less
applicability	makes "everything" run faster	does not apply to some problems
improvement	quantitative improvements	dramatic qualitative improvements possible