PROLOGUE
THE EXPONENTIAL FUNCTION

This is the most important function in mathematics. It is defined, for every complex number z, by the formula

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1}$$

The series (1) converges absolutely for every z and converges uniformly on every bounded subset of the complex plane. Thus exp is a continuous function. The absolute convergence of (1) shows that the computation

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{m=0}^{\infty} \frac{b^m}{m!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k} = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$$

is correct. It gives the important addition formula

$$\exp(a)\exp(b) = \exp(a+b), \tag{2}$$

valid for all complex numbers a and b.

We define the number e to be $\exp(1)$, and shall usually replace $\exp(z)$ by the customary shorter expression e^z. Note that $e^0 = \exp(0) = 1$, by (1).

Theorem

(*a*) *For every complex z we have $e^z \neq 0$.*
(*b*) exp *is its own derivative:* $\exp'(z) = \exp(z)$.

(*c*) *The restriction of* exp *to the real axis is a monotonically increasing positive function, and*

$$e^x \to \infty \text{ as } x \to \infty, \qquad e^x \to 0 \text{ as } x \to -\infty.$$

(*d*) *There exists a positive number* π *such that* $e^{\pi i/2} = i$ *and such that* $e^z = 1$ *if and only if* $z/(2\pi i)$ *is an integer.*
(*e*) exp *is a periodic function, with period* $2\pi i$.
(*f*) *The mapping* $t \to e^{it}$ *maps the real axis onto the unit circle.*
(*g*) *If* w *is a complex number and* $w \neq 0$, *then* $w = e^z$ *for some* z.

PROOF By (2), $e^z \cdot e^{-z} = e^{z-z} = e^0 = 1$. This implies (*a*). Next,

$$\exp'(z) = \lim_{h\to 0} \frac{\exp(z+h) - \exp(z)}{h} = \exp(z) \lim_{h\to 0} \frac{\exp(h) - 1}{h} = \exp(z).$$

The first of the above equalities is a matter of definition, the second follows from (2), and the third from (1), and (*b*) is proved.

That exp is monotonically increasing on the positive real axis, and that $e^x \to \infty$ as $x \to \infty$, is clear from (1). The other assertions of (*c*) are consequences of $e^x \cdot e^{-x} = 1$.

For any real number t, (1) shows that e^{-it} is the complex conjugate of e^{it}. Thus

$$|e^{it}|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it} = e^{it-it} = e^0 = 1,$$

or

$$|e^{it}| = 1 \qquad (t \text{ real}). \tag{3}$$

In other words, if t is real, e^{it} lies on the unit circle. We define $\cos t$, $\sin t$ to be the real and imaginary parts of e^{it}:

$$\cos t = \text{Re}\,[e^{it}], \qquad \sin t = \text{Im}\,[e^{it}] \qquad (t \text{ real}). \tag{4}$$

If we differentiate both sides of Euler's identity

$$e^{it} = \cos t + i \sin t, \tag{5}$$

which is equivalent to (4), and if we apply (*b*), we obtain

$$\cos' t + i \sin' t = ie^{it} = -\sin t + i \cos t,$$

so that

$$\cos' = -\sin, \qquad \sin' = \cos. \tag{6}$$

The power series (1) yields the representation

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots. \tag{7}$$

Take $t = 2$. The terms of the series (7) then decrease in absolute value (except for the first one) and their signs alternate. Hence cos 2 is less than the sum of the first three terms of (7), with $t = 2$; thus $\cos 2 < -\frac{1}{3}$. Since $\cos 0 = 1$ and cos is a continuous real function on the real axis, we conclude that there is a smallest positive number t_0 for which $\cos t_0 = 0$. We define

$$\pi = 2t_0 . \tag{8}$$

It follows from (3) and (5) that $\sin t_0 = \pm 1$. Since

$$\sin' (t) = \cos t > 0$$

on the segment $(0, t_0)$ and since $\sin 0 = 0$, we have $\sin t_0 > 0$, hence $\sin t_0 = 1$, and therefore

$$e^{\pi i/2} = i. \tag{9}$$

It follows that $e^{\pi i} = i^2 = -1$, $e^{2\pi i} = (-1)^2 = 1$, and then $e^{2\pi i n} = 1$ for every integer n. Also, (e) follows immediately:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z. \tag{10}$$

If $z = x + iy$, x and y real, then $e^z = e^x e^{iy}$; hence $|e^z| = e^x$. If $e^z = 1$, we therefore must have $e^x = 1$, so that $x = 0$; to prove that $y/2\pi$ must be an integer, it is enough to show that $e^{iy} \neq 1$ if $0 < y < 2\pi$, by (10).

Suppose $0 < y < 2\pi$, and

$$e^{iy/4} = u + iv \qquad (u \text{ and } v \text{ real}). \tag{11}$$

Since $0 < y/4 < \pi/2$, we have $u > 0$ and $v > 0$. Also

$$e^{iy} = (u + iv)^4 = u^4 - 6u^2v^2 + v^4 + 4iuv(u^2 - v^2). \tag{12}$$

The right side of (12) is real only if $u^2 = v^2$; since $u^2 + v^2 = 1$, this happens only when $u^2 = v^2 = \frac{1}{2}$, and then (12) shows that

$$e^{iy} = -1 \neq 1.$$

This completes the proof of (d).

We already know that $t \to e^{it}$ maps the real axis *into* the unit circle. To prove (f), fix w so that $|w| = 1$; we shall show that $w = e^{it}$ for some real t. Write $w = u + iv$, u and v real, and suppose first that $u \geq 0$ and $v \geq 0$. Since $u \leq 1$, the definition of π shows that there exists a t, $0 \leq t \leq \pi/2$, such that $\cos t = u$; then $\sin^2 t = 1 - u^2 = v^2$, and since $\sin t \geq 0$ if $0 \leq t \leq \pi/2$, we have $\sin t = v$. Thus $w = e^{it}$.

If $u < 0$ and $v \geq 0$, the preceding conditions are satisfied by $-iw$. Hence $-iw = e^{it}$ for some real t, and $w = e^{i(t+\pi/2)}$. Finally, if $v < 0$, the preceding two cases show that $-w = e^{it}$ for some real t, hence $w = e^{i(t+\pi)}$. This completes the proof of (f).

If $w \neq 0$, put $\alpha = w/|w|$. Then $w = |w|\alpha$. By (c), there is a real x such that $|w| = e^x$. Since $|\alpha| = 1$, (f) shows that $\alpha = e^{iy}$ for some real y. Hence $w = e^{x+iy}$. This proves (g) and completes the theorem. ////

We shall encounter the integral of $(1 + x^2)^{-1}$ over the real line. To evaluate it, put $\varphi(t) = \sin t/\cos t$ in $(-\pi/2, \pi/2)$. By (6), $\varphi' = 1 + \varphi^2$. Hence φ is a monotonically increasing mapping of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, and we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\pi/2}^{\pi/2} \frac{\varphi'(t)\, dt}{1 + \varphi^2(t)} = \int_{-\pi/2}^{\pi/2} dt = \pi.$$